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Analyticity for One-Dimensional Systems with Long-Range Superstable Interactions

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We consider unbounded spin systems and classical continuous particle systems in one dimension. We assume that the interaction is described by a superstable two-body potential with a decay at large distances at least as $r^{-2}(\ln r)^{-(2+\epsilon)}$, $\epsilon > 0$. We prove the analyticity of the free energy and of the correlations as functions of the interaction parameters. This is done by using a "renormalization group technique" to transform the original model into another, physically equivalent, model which is in the high-temperature (small-coupling) region.

KEY WORDS: One-dimensional Gibbs systems, transfer matrix, Markov chains, renormalization group, decimation procedure, cluster expansion.

INTRODUCTION

For one-dimensional lattice spin models and systems of classical particles on a line, it is expected that the free energy and the correlation functions are analytic functions of the interaction parameters, if the potential decays fast enough with the distance.(1) A number of papers have been devoted to this subject(2–7) and the analyticity has been proven for a large class of models. This class however does not include certain interesting cases, such as unbounded spin systems, or systems of particles without hard core, with

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a two-body interaction decaying asymptotically with the distance r as $r^{-(2+\epsilon)}$, $\epsilon > 0$.[5]

In this paper we are able to deal with such systems. We assume that the Hamiltonian satisfies the conditions that are needed for the validity of the well-known superstability estimates on the correlation functions.[9,10] The proof of the analyticity of the free energy for a one-dimensional lattice gas interacting via long-range many-body potential was given in Ref. 7. Here we extend the result to systems of unbounded spins and classical particles without hard core, when the potential decays at large distances at least as fast as $r^{-2}(\ln r)^{-2+\epsilon}$ and a superstability condition on the interaction is assumed.

In Ref. 7 and in the present paper a renormalization group technique is used. The lattice is partitioned into blocks, and a suitable decimation procedure is performed over the block variables. It is then shown that, in the resulting block system, blocks are in fact weakly interacting, so that a standard way to prove the analyticity of the free energy applies. The system is transformed into a so-called polymer model, whose properties are then investigated by means of a cluster expansion.[12,13] As a consequence of the weak coupling of a block system, the corresponding polymer model is in fact a gas with low activity. In Ref. 7 the boundedness of the interaction energy among two semiinfinite subsystems is exploited to show that blocks are weakly coupled. We remark that this requirement is a common feature of most of the previous papers on this subject. For unbounded spin systems and systems of particles without hard core this condition is not satisfied, as the interaction energy among two half-lines can be arbitrarily large if the spin values, or the density of particles, are large enough. Therefore the weak coupling among blocks can be expected to hold only in a subset of the set of block configurations. We use the superstability estimates to overcome the difficulties arising from this fact. We show that block configurations giving rise to large interaction energy contribute in a negligible way to the free energy.

In carrying out our argument we prove a theorem on the exponential approach to equilibrium for Markov chains satisfying a weaker hypothesis than the Doeblin condition. We think that this theorem has some interest on its own and could be applicable in other situations.

In the following section we describe our models, state the main results, and give a heuristic sketch of the proofs, underlining the main technical difficulties. A rigorous formulation of the ideas of Section 1 together with

[5] Note: It is known[8] that, also for the compact spin case, the free energy can be not analytic if the interaction decays only as r^{-2}, but analyticity is expected to hold for interactions decaying faster.

detailed proofs are given in Sections 2 and 3. In the concluding remarks we discuss some possible extensions of our results.

1. SECTION 1

To each site of a one-dimensional lattice $\mathbb{Z}$ we associate a spin variable taking values in a topological space $\mathfrak{X}$, e.g., on the real line.

A *spin configuration* in a region $\Lambda \subset \mathbb{Z}$ is a function $s_\Lambda : \Lambda \to \mathfrak{X}^\Lambda$. Its value at $x \in \mathbb{Z}$ will be denoted by s_x. For any nonoverlapping regions $\Delta_1, \Delta_2 \subset \mathbb{Z}$, $\Delta_1 \cap \Delta_2 = \phi$, we shall denote by $s_{\Delta_1} \vee s_{\Delta_2}$ the spin configuration in $\Delta_1 \cup \Delta_2$:

$$s_{\Delta_1} \vee s_{\Delta_2}|_{\Delta_1} = s_{\Delta_1}, \qquad s_{\Delta_1} \vee s_{\Delta_2}|_{\Delta_2} = s_{\Delta_2}$$

The *energy* of a configuration in $\Lambda \subset \mathbb{Z}$ will be a function from $\mathfrak{X}^\Lambda$ to $\mathbb{R}$ of the form

$$H_\Lambda(s_\Lambda) = \sum_{x \in \Lambda} \varphi(s_x) + \sum_{\substack{x, y \in \Lambda \\ x < y}} J_{|x-y|}(s_x, s_y) \tag{1.1}$$

where φ and J_r, $r \in \mathbb{Z}^+$, are real-valued continuous functions on $\mathfrak{X}, \mathfrak{X}^2$, respectively. The *free measure* ν is a positive measure on $\mathfrak{X}$, not identically zero. We shall use the notation $\nu_\Lambda(ds_\Lambda) = \prod_{x \in \Lambda} \nu(ds_x)$, $|\Lambda| = \sharp(\Lambda)$ for $\Lambda \subset \mathbb{Z}$. The following two cases will be considered:

1.1. Lattice Spin Models

In this case $\mathfrak{X} = \mathbb{R}$, ν is a measure such that

$$\int \nu(ds) e^{-as^2} < \infty, \qquad \text{if} \quad a > 0 \tag{1.2}$$

and the one- and two-body interactions φ, J_r are continuous functions. We shall assume the following.

(i) There exists a positive, nondecreasing real function F on $\mathbb{Z}^+$ such that[6]

$$\begin{gathered} |J_r(s, s')| \leqslant |s|\,|s'| / \left[r^2 \ln(r+1) F(r) \right] \\ \sum_{r \in \mathbb{Z}^+} \left[rF(r) \right]^{-1} < \infty \end{gathered} \tag{1.3}$$

[6] Note: We might as well assume the more general condition

$$|J_r(s, s')| \leqslant (|s|\,|s'| + K) / \left[r^2 \ln(r+1) F(r) \right]$$

for some positive constant K. All our arguments apply to this case.

(ii) There exists $A > 0$, $\delta \in \mathbb{R}$ such that, if Λ is any finite subset of $\mathbb{Z}$,

$$H_\Lambda(s_\Lambda) \geqslant \sum_{x \in \Lambda} \left(A|s_x|^2 - \delta \right) \tag{1.4}$$

We shall refer to (1.4) as to the superstability condition.

1.2. Systems of Classical Particles

In this case $\mathfrak{X} = \bigcup_{n=0}^{\infty} \mathfrak{X}_n$, where, given a positive ρ_0, $\mathfrak{X}_0$ describes the configuration in which the interval $[0, \rho_0]$ is empty and $\mathfrak{X}_n = [0, \rho_0]^n$ is the set of configurations with n particles in the interval $[0, \rho_0]$. We define $|s| = n$ if the spin variable $s \in \mathfrak{X}_n$. A sequence $s^{(k)}$ in $\mathfrak{X}$ is said to converge to $s \in \mathfrak{X}$ if there is k_0 such that for $k > k_0$, $|s^{(k)}| = |s|$ and the sequence $s^{(k)}$, $k > k_0$, converges to s in the usual topology of $\mathfrak{X}_{|s|}$.

ν is the measure $\nu = \bigoplus_{n=0}^{\infty} \nu_n$ where $\nu_0(\mathfrak{X}_0) = 1$ and $n!\,\nu_n$ is the Lebesgue measure on $\mathfrak{X}_n$.

The one- and two-body interaction are of the form

$$\begin{aligned} \varphi(s) &= \mu|s| + \sum_{\substack{i,j=1 \\ i<j}}^{n} v(|\xi_i - \xi_j|) \\ J_{|x-y|}(s, s') &= \sum_{i=1}^{n} \sum_{j=1}^{m} v(|\xi_i - \eta_j + (x-y)\rho_0|) \end{aligned} \tag{1.5}$$

if $|s|, |s'| \neq 0$, $s = (\xi_1, \ldots, \xi_n)$, $s' = (\eta_1, \ldots, \eta_m)$ $\varphi(s) = J_r(s, s') = J_r(s', s) = 0$ if $|s| = 0$ where $\mu \in \mathbb{R}$, $z = e^{\mu}$ is the fugacity and v is a real, continuous function on $\mathbb{R}$.

We shall assume the following.

(i) $|v(r)|$ is bounded by a nonincreasing $\tilde{v}(r)$ such that

$$\int_0^{\infty} \tilde{v}(r) r \ln(r+1)\, dr < \infty \tag{1.6}$$

We remark that, by choosing

$$F(r) = \left\{ r^2 \ln(r+1) \tilde{v}\left[\rho_0(r-1) \right] \right\}^{-1}$$

condition (1.3) is satisfied.

(ii) the superstability condition (1.4) is satisfied.

We shall consider the following set of complex perturbations of the Hamiltonian:

$$V_\Lambda(\omega, s_\Lambda) = \sum_{i=1}^{d} \omega_i \left[\sum_{x \in \Lambda} \psi^i(s_x) + \sum_{\substack{x, y \in \Lambda \\ x<y}} G^i_{|x-y|}(s_x, s_y) \right] \tag{1.7}$$

where $\omega_i \in \mathbb{C}$, ψ^i, G_r^i are real-valued measurable functions on $\mathfrak{X}, \mathfrak{X}^2$ such that

$$\text{(i)} \qquad |G_r^i(s,s')| \leqslant |s|\,|s'|/\left[r^2 \ln(r+1)\bar{F}(r)\right] \tag{1.8}$$

where $\bar{F}$ is a real function, nondecreasing, and

$$\sum_{r\in\mathbb{Z}^+} \left[r\bar{F}(r)\right]^{-1} < \infty$$

(ii) there exist $\bar{A} > 0$, $\bar{\delta} \in \mathbb{R}$ such that, if $\omega = \max_{1\leqslant i\leqslant d}|\omega_i|$, $\Lambda \subset \mathbb{Z}$

$$|V_\Lambda(\omega, s_\Lambda)| \leqslant \omega \sum_{x\in\Lambda} \left(\bar{A}|s_x|^2 + \bar{\delta}\right) \tag{1.9}$$

The partition function with empty boundary conditions corresponding to the complex Hamiltonian $H_\Lambda(\omega,\cdot) = H_\Lambda(\cdot) + V_\Lambda(\omega,\cdot)$ is then defined as

$$Z_\Lambda(\omega) = \int \nu_\Lambda(ds_\Lambda) \exp\left[-H_\Lambda(\omega, s_\Lambda)\right] \tag{1.10}$$

The main result of this paper can now be stated as follows:

Theorem 1.1. Let $Z_\Lambda(\omega)$ be defined as in (1.10) and assume that (1.2), (1.3), (1.4), or (1.6), (1.4) be satisfied. Then there exists a sphere $\Sigma \in \mathbb{C}^d$, centered at the origin, such that

$$\mathscr{F}(\omega) = \lim_{\Lambda \nearrow \mathbb{Z}} \ln\left[Z_\Lambda(\omega)\right]/|\Lambda| \tag{1.11}$$

exists and is a holomorphic function of ω on Σ.

We describe now the spin block model. Let L, n, p be positive integers, L odd. Λ_p is the interval of $\mathbb{Z}$, centered at the origin, that contains $|\Lambda_p| = (2p+1)L + 2pnL$ points. Consider the decomposition of Λ_p into consecutive blocks:

$$\Lambda_p = A_{-p} \cup B_{-p} \cup A_{-p+1} \cup \cdots \cup B_{-1} \cup A_0 \cup B_0 \cup \cdots \cup B_{p-1} \cup A_p$$

where $|A_i| = L$ and $|B_i| = nL$. We shall also consider the decomposition of B_i into n consecutive blocks of length L:

$$B_i = \bigcup_{k=1}^{n} B_{i,k}, \qquad |B_{i,k}| = L$$

In the following, in order to simplify the notation, we shall denote the spin block configurations $s_{A_i}, s_{B_i}, s_{B_{i,k}}$ by α_i, β_i, $\beta_{i,k}$, respectively.

Given L, we decompose the interaction into a short-range term and a tail, that will be dealt with as a perturbation:

$$J_r(s,s') = J_r^{<L}(s,s') + J_r^{>L}(s,s') \tag{1.12}$$

where $J_r^{<L} = J_r$, $1 \leqslant r \leqslant L$, $J_r^{<L} = 0$, $r > L$ for the (a) model and, for the

(b) model:

$$J_{|x-y|}^{<L}(s,s') = \sum_{i=1}^{n} \sum_{j=1}^{m} v^{<L}\left[\,|\xi_i - \eta_j + (x-y)\rho_0|\,\right] \tag{1.13}$$

where, given $0 < \epsilon < \rho_0$, $v^{<L}$ is a continuous function $v^{<L}(r) = \sigma(r)v(r)$, with $0 \leqslant \sigma(r) \leqslant 1$, $\sigma(r) = 1$ for $0 \leqslant r < L - \epsilon$, $\sigma(r) = 0$ for $r \geqslant L$.

The short-range term in the Hamiltonian will be denoted by

$$H_\Lambda^{<L}(s_\Lambda) = \sum_{x\in\Lambda} \varphi(s_x) + \sum_{\substack{x,\,y\in\Lambda\\ x<y}} J_{|x-y|}^{<L}(s_x, s_y) \tag{1.14}$$

For technical reasons we shall put a cutoff on the spin values in the block model. Let N be a positive number. We shall denote by $\mathfrak{X}_{\leqslant N}$ the set $\mathfrak{X}_{\leqslant N} = \{s \in \mathfrak{X} : |s| \leqslant N\}$, by χ^N its characteristic function and, for any $\Lambda \subset \mathbb{Z}$

$$\chi_\Lambda^N(s_\Lambda) = \prod_{x\in\Lambda} \chi^N(s_x), \qquad \nu_\Lambda^N(ds_\Lambda) = \nu_\Lambda(ds_\Lambda)\chi_\Lambda^N(s_\Lambda)$$

We define the "transfer matrix" $T_{N,L}(s,s')$ for $s, s' \in (\mathfrak{X}_{\leqslant N})^L$ by

$$T_{N,L}(s,s') = \exp\left[-h_L(s)/2 - W_L(s,s') - h_L(s')/2\right] \tag{1.15}$$

where

$$\begin{gathered} h_L(s) = H_{[1,L]}^{<L}(s) \\ W_L(s,s') = \sum_{i,\,j=1}^{L} J_{|i-j-L|}^{<L}(s_i, s_j') \end{gathered} \tag{1.16}$$

and its iterates by

$$T_{N,L}^{k+1}(s,s') = \int \nu_L^N(ds'')\,T_{N,L}^k(s,s'')\,T_{N,L}(s'',s') \tag{1.17}$$

where $\nu_L^N = \nu_{[1,L]}^N$. It follows from the previous definitions that

$$T_{N,L}(s,s') = T_{N,L}(Rs', Rs) \tag{1.18}$$

where the reversed configuration Rs is defined by $(Rs)_i = s_{L-i}$ for (a) models and $(Rs)_i = \boldsymbol{\rho}_0 - s_{L-i}$ for (b) models, where $\boldsymbol{\rho}_0$ is the vector with s_{L-i} components all equal to ρ_0. We denote by $T_{N,L}$ the operator on the Banach space $\mathscr{C}((\mathfrak{X}_{\leqslant N})^L)$:

$$(T_{N,L}u)(s) = \int \nu_L^N(ds')\,T_{N,L}(s,s')\,u(s') \tag{1.19}$$

The following properties, which will be used in Sections 2 and 3, are proven in Appendix A.

P.1. $T_{N,L}$ has an eigenvalue $\lambda_{N,L} > 0$, that exceeds in modulus all the other points in the spectrum, and corresponding unique left and right eigenvectors $\tilde{v}_{N,L}, v_{N,L}$. $\tilde{v}_{N,L}(s) = v_{N,L}(Rs)$, $v_{N,L}(s) > 0$.

We shall always assume the normalization

$$\int \nu_L^N(ds)\,\hat{v}_{N,L}(s) v_{N,L}(s) = 1 \tag{1.20}$$

P.2. There exists L_0 such that for $L > L_0$, $\lambda_L = \lim_{N\to\infty}\lambda_{N,L}$ exists, and it satisfies $\lambda_L \leqslant (C)^L$, where C is a constant.

P.3. Superstability estimates. Let μ be a measure on $(\mathfrak{X}_{\leqslant N})^{kL}$ absolutely continuous with respect to $\nu^N_{[1,kL]}$ with density given by

$$\frac{d\mu}{d\nu^N_{[1,kL]}}(S_{[1,kL]}) = \frac{\tilde{u}(s^{(1)}) T_{N,L}(s^{(1)}, s^{(2)}) \ldots T_{N,L}(S^{(k-1)}, s^{(k)}) u(s^{(k)})}{\lambda_{N,L}^{k-1}} \tag{1.21}$$

where $s^{(i)} = s_{[1,kL]}|_{[(i-1)L+1,iL]}$, $\tilde{u} \in \{\tilde{v}_{N,L}, \exp(-h_L/2)\lambda^{-1/2}\}$, $u \in \{v_{N,L}, \exp(-h_L/2)\lambda^{-1/2}\}$. There exist N_0, L_0, A^*, δ^* such that, if $N > N_0$, $L > L_0$, $\Delta \subset [1,kL]$:

$$\int \nu^N_{[1,kL]\setminus\Delta}(ds_{[1,kL]\setminus\Delta}) \frac{d\mu}{d\nu^N_{[1,kL]}}(s_{[1,kL]}) / \mu\big((\mathfrak{X}_{\leqslant N})^{kL}\big) \leqslant \exp\Big[-\sum_{x\in\Delta}(A^*|s_x|^2 - \delta^*)\Big] \tag{1.22}$$

We are now able to describe the strategy of the proof of the theorem. Given L, n, p, N the partition function of the block model in the volume Λ_p, with empty boundary conditions, can be written in the form

$$Z_{\Lambda_p,N}(\omega) = \int \nu^N_{\Lambda_p}(ds_{\Lambda_p}) \exp\big[-h_L(\alpha_{-p})/2 - h_L(\alpha_p)/2\big] \times \prod_{i=-p}^{p-1} T_{N,L}(\alpha_i, \beta_i, \alpha_{i+1}) \exp\big[W^\omega(S_{\Lambda_p})\big] \tag{1.23}$$

where

$$T_{N,L}(\alpha_i, \beta_i, \alpha_{i+1}) = T_{N,L}(\alpha_i, \beta_{i,1}) \prod_{k=1}^{n-1} T_{N,L}(\beta_{i,k}, \beta_{i,u+1}) T_{N,L}(\beta_{i,n}, \alpha_{i+1})$$

so that the complex interactions and the interactions due to the tail of the potential only appear in W^ω.

Let now $\mathscr{S}^L_M$ be the subset of the configurations

$$\mathscr{S}^L_M = \big\{s \in \mathfrak{X}^L : |s_i| \leqslant Ml(i), i = 1, \ldots, L\big\} \tag{1.24}$$

where $l(i) = \{\min[\ln(i+1), \ln(L-i+2)]\}^{1/2}$. For any fixed M, it follows from (1.3), (1.6), (1.8) that all the terms contributing to $W^{\omega}(s_{\Lambda_p})$ can be made arbitrarily small if ω is small, L big enough, provided that all the block configurations are in $\mathscr{S}_M^L$. In the following sections it will be shown, using the superstability estimates, that most of the statistical weight is in fact on the configurations in $\mathscr{S}_M^L$, so that the block model, with partition function (1.23), can be dealt with as a perturbation of a block model in which the only interacting blocks are the nearest neighbors. If we perform then the "decimation procedure" (average over the B block configurations), we reduce the partition function of this latter model to the form

$$\tilde{Z}_{\Lambda_p,N} = \int \nu_{\Lambda_p}^N(ds_{\Lambda_p}) \exp\left(H_{\Lambda_p}^{<L}(s_{\Lambda_p})\right)$$
$$= \int \prod_{i=-p}^{p} \nu_L^N(\alpha_i) \exp\left[-h(\alpha_{-p})/2 - h(\alpha_p)/2\right] \prod_{i=-p}^{p-1} T_{N,L}^{n+1}(\alpha_i, \alpha_{i+1}) \tag{1.25}$$

This model, however, is still nontrivial. We must show that distinct A blocks are weakly coupled, in order to perform a cluster expansion. We shall prove the following estimate:

$$|T_{N,L}^n(s,s') / \left[v_{N,L}(s) \tilde{v}_{N,L}(s') \lambda_{N,L}^n \right] - 1| \leqslant \exp\left(\bar{c}_1 M^2 - n e^{-\bar{c}_2 M^2}\right) \quad \text{if} \quad s, s' \in \mathscr{S}_M^L \tag{1.26}$$

that gives the weak coupling for n big enough. The estimate will be obtained by computing the rate of exponential approach to equilibrium of a Markov chain with state space $(\mathfrak{X}_{\leqslant N})^L$, endowed with the measure ν_L^N, transition kernel:

$$P_{s,s'} = T_{N,L}(s,s') v_{N,L}(s) / \left[\tilde{v}_{N,L}(s') \lambda_{N,L} \right] \tag{1.27}$$

and unique equilibrium measure (see P.1) $v_{N,L} \tilde{v}_{N,L} \nu_L^N$. We remark that we shall need estimates that are independent of the spin cutoff N and of the block size L, so that this computation is not trivial. In fact, as far as we know, there are no results in the literature allowing to prove that, in our case, the rate of exponential approach to equilibrium has a positive, uniform lower bound. We remark that, for example, Doeblin condition[15] is not uniformly satisfied under our hypotheses.

2. SECTION 2

In this section we shall prove an estimate on the approach to the equilibrium of the iterates of the transfer operator. Throughout this section we shall keep the values of N and L fixed and we shall use the following abbreviated notation:

$$\begin{gathered} T^k(s,s') = T^k_{N,L}(s,s'), \qquad k \geqslant 1 \\ v(s) = v_{N,L}(s), \qquad \tilde{v}(s) = \tilde{v}_{N,L}(s), \qquad \lambda = \lambda_{N,L} \\ h(s) = h_L(s), \qquad W(s,s') = W_L(s,s') \\ \mathscr{S}_M = \mathscr{S}^L_M, \qquad \rho(ds) = \nu^N_L(ds) \end{gathered} \tag{2.1}$$

Our main result is the following theorem:

Theorem 2.1. There exist positive integers N_0, L_0 and positive constants $\bar{c}_1$ and $\bar{c}_2$ such that for $N > N_0$, $L > L_0$, and $M > 1$,

$$|T^n(s,s')/[\lambda^n v(s)\tilde{v}(s')] - 1| \leqslant \exp(\bar{c}_1 M^2 - ne^{-\bar{c}_2 M^2}) \qquad \forall s,s' \in \mathscr{S}_M \tag{2.2}$$

The crucial point in (2.2) is the independence of the right-hand side from N and L. Estimates depending on these parameters are quite easy to obtain.

Our method is based on the introduction of a suitable Markov chain and on the study of its approach to the equilibrium.

First we need an estimate on v and $\tilde{v}$. The probability measures proportional to $\exp[-h(s)/2]v(s)\rho(ds)$ and $v(s)\tilde{v}(s)\rho(ds)$ are, respectively, the restrictions to the zeroth L block of the semiinfinite and infinite Gibbs measures corresponding to the potential truncated at distance L (see Appendix A). In the case of one-dimensional compact spin systems with interaction decaying sufficiently fast it is well known (see, e.g., Refs. 16 or 17) that these two measures are mutually absolutely continuous with Radon–Nykodim derivative bounded by a constant. The following proposition is an extension of this result to the unbounded spin case.

Proposition 2.2. There exist positive constants c_{10} and c_{11} such that for $N > N_0$, $L > L_0$, and $M > 1$:

$$c_{10}^{-1} e^{-c_{11}M^2} \leqslant v(s)\lambda^{1/2}/\exp[-h(s)/2] \leqslant c_{10} e^{c_{11}M^2} \tag{2.3}$$

for every $s \in \mathscr{S}_M$, where the positive constants N_0 and L_0 are introduced in Appendix A. The same estimate is of course true for $\tilde{v}(s)$.

Proof. Let us consider a probability measure μ on $\mathfrak{X}^L$ satisfying superstability estimates with constants A^* and δ^*. From the definition

(1.24) we have that for M large enough

$$\mu(\mathfrak{X}^L\backslash\mathscr{S}_M) \leqslant \sum_{i=1}^{L} \mu(\{s : |s_i| \geqslant Ml(i)\})$$

$$\leqslant 2\sum_{i=1}^{\infty} c_1\exp\left[-A^*\ln(i+1)M^2/2 + \delta^*\right] \leqslant c_2e^{-c_3M^2} \quad (2.4)$$

where c_1, c_2, and c_3 are positive constants depending only on A^* and δ^*. Let now $s \in \mathscr{S}_M$ and let us denote by $\hat{\chi}_r$ the characteristic function of the interval $[0, r]$. We have, for θ large enough:

$$\int \mu\,(ds')\exp\left[-W(s,s')\right]$$

$$= \sum_{\Delta\subset[1,L]} \int \mu\,(ds')\exp\left[-W(s,s')\right] \prod_{i\in\Delta} \chi_{\theta M[\ln(i+1)]^{1/2}}(|s_i'|)$$

$$\times \prod_{j\in[1,L]\backslash\Delta} (1 - \chi_{\theta M[\ln(j+1)]^{1/2}}(|s_j'|))$$

$$\leqslant \sum_{\Delta\subset[1,L]} \prod_{i\in\Delta} \exp\left\{\sum_{k>0} f(i+k)\left[\ln(i+1)\ln(k+2)\right]^{1/2}\theta^2M^2\right\}$$

$$\times \prod_{j\in[1,L]\backslash\Delta} \int_{|s_j'|>\theta M[\ln(j+1)]^{1/2}} \nu\,(ds_j')$$

$$\times \exp\left\{\sum_{k>0} f(j+k)\theta M\left[\ln(K+2)\right]^{1/2}|s_j'| - A^*|s_j'|^2 + \delta^*\right\}$$

$$\leqslant \exp\left\{\sum_{\substack{i>0\\ j>0}} f(i+j)\theta M^2\left[\ln(i+2)\ln(j+2)\right]^{1/2}\right\}$$

$$\times \prod_{i>0} \left\{1 + c_4\exp\left[-A^*\theta^2M^2\ln(i+1)\right]\right\}$$

$$\leqslant c_5e^{c_6M^2} \quad (2.5)$$

where $f(r) = [r^2\ln(r+1)F(r)]^{-1}$ [see (1.3)].

c_4, c_5, and c_6, as well as all the constants c_i of this section, are positive constants independent of N and L, for $N > N_0$ and $L > L_0$.

Now we shall give an estimate on the eigenvectors. From (1.23), (2.4) and (2.5) we get for $s, s' \in \mathscr{S}_M$, $N > N_0$, $L > L_0$.

$$\begin{aligned}
&v(s)\exp\left[h(s)/2\right]/\left\{v(s')\exp\left[h(s')/2\right]\right\} \\
&\quad = \int \rho(dt)\exp\left[-W(s,t) - h(t)/2\right]v(t) \\
&\qquad \div \left\{\int \rho(dt)\exp\left[-W(s',t) - h(t)/2\right]v(t)\right\} \\
&\quad \geqslant c_5^{-1}e^{-c_6M^2}\int_{\mathscr{S}_M} \rho(dt) \\
&\qquad \times \exp\left[-W(s,t) - h(t)/2\right]v(t)/\left\{\int \rho(dt)\exp\left[-h(t)/2\right]v(t)\right\} \\
&\quad \geqslant c_5^{-1}e^{-c_6M^2}\left(1 - c_2e^{-c_3M^2}\right) \\
&\qquad \times \exp\left(-\sum_{\substack{i>0\\ j>0}} f(i+j)\left[\ln(i+2)\ln(j+2)\right]^{1/2}M^2\right) \\
&\quad \geqslant c_7^{-1}e^{-c_8M^2}
\end{aligned} \tag{2.6}$$

If M is large enough, independently of N and L, it follows from superstability estimates and from the estimate given by (2.4) that

$$\begin{aligned}
&\text{(i)} \qquad \frac{3}{4} \leqslant \int_{\mathscr{S}_M} \rho(ds)\,\tilde{v}(s)v(s) \\
&\text{(ii)} \qquad \frac{1}{2} \leqslant \int_{\mathscr{S}_M \times \mathscr{S}_M} \rho(ds)\,\rho(ds')\,\tilde{v}(s)v(s')\lambda^{-1} \\
&\qquad\qquad \times \exp\left[-h(s)/2 - W(s,s') - h(s')/2\right] \leqslant 1
\end{aligned} \tag{2.7}$$

On the other hand, it follows from (2.6) that there exists a constant E such that for $s \in \mathscr{S}_M$

$$Ec_7^{-1}e^{-c_8M^2}\lambda^{-1/2}\exp\left[-h(s)/2\right] \leqslant v(s) \leqslant Ec_7e^{c_8M^2}\lambda^{-1/2}\exp\left[-h(s)/2\right] \tag{2.8}$$

By inserting the above inequalities in (2.7i) and by using (2.7ii), we get

$$\begin{aligned}
E &\leqslant \left(2c_7^2e^{2c_8M^2}e^{c_9M^2}\right)^{1/2} \\
E &\geqslant \left[\left(\tfrac{3}{4}\right)^2c_7^{-2}e^{-2c_8M^2}e^{-c_9M^2}\right]^{1/2}
\end{aligned} \tag{2.9}$$

where $c_9 = \sum_{i>0,\, j>0} f(i+j)[\ln(i+2)\ln(j+2)]^{1/2}$ and by inserting (2.9) in (2.8) we finally get (2.3).

From now on we shall study the properties of approach to the equilibrium of the Markov chain introduced in Section 1.

Proposition 2.3. There exists a positive constant c_{12} such that for $N > N_0$, $L > L_0$, $M > 1$, every positive integer n and every $s, s' \in \mathscr{S}_M$:

$$P^n_{s,s'}/v(s')\tilde{v}(s') \leqslant e^{-c_{12}M^2} \tag{2.10}$$

where P^n is the nth iterate of the stochastic kernel (1.27).

Proof. We have

$$\begin{aligned}
P^n_{s,s'}/v(s')\tilde{v}(s') &= T^n(s,s')/\lambda^n v(s)\tilde{v}(s') \\
&\leqslant c_{10}^2 e^{2c_{11}M^2}\int \rho(dt)\rho(dt')\lambda^{-n+1}\exp\left[-W(s,t)-h(t)/2\right] \\
&\quad \times T^{n-2}(t,t')\exp\left[-h(t')/2-W(t',s')\right] \\
&= c_{10}^2 e^{2c_{11}M^2}\int \rho(dt)\rho(dt')\exp\left[-W(s,t)-h(t)/2\right] \\
&\quad \times T^{n-2}(t,t')\exp\left[-h(t')/2-W(t',s')\right] \\
&\quad \div \int \rho(dt)\rho(dt')\exp\left[-h(t)/2\right]T^{n-2}(t,t')\exp\left[-h(t')/2\right] \\
&\quad \times \int \rho(dt)\rho(dt')\exp\left[-h(t)/2\right]T^{n-2}(t,t')\exp\left[-h(t')/2\right] \\
&\quad \div \int \rho(dt)\rho(dt')\tilde{v}(t)T^{n-2}(t,t')v(t') \\
&\leqslant c_{10}^2 e^{2c_{11}M^2}c_5^2 e^{2c_6M^2}2\int_{\mathscr{S}_M\times\mathscr{S}_M}\rho(dt)\rho(dt') \\
&\quad \times \exp\left[-h(t)/2\right]T^{n-2}(t,t')\exp\left[-h(t')/2\right] \\
&\quad \div \lambda\int_{\mathscr{S}_M\times\mathscr{S}_M}\rho(dt)\rho(dt')\tilde{v}(t)T^{n-2}(t,t')v(t') \\
&\leqslant 2c_{10}^2 e^{2c_{11}M^2}c_5^2 e^{2c_6M^2}\left\{\sup_{t\in\mathscr{S}_M}\exp\left[-h(t)/2\right]/\lambda^{1/2}v(t)\right\}^2 \\
&\leqslant 2c_{10}^4 e^{4c_{11}M^2}c_5^2 e^{2c_6M^2} \leqslant e^{c_{12}M^2}
\end{aligned} \tag{2.11}$$

where we have used the result and the method of proof of Proposition 2.2 [see Eq. (2.5)] and superstability estimates for the probability measure proportional to $\rho(dt)\rho(dt')\exp[-h(t)/2-h(t')/2]T^{n-2}(t,t')$.

The following is an adaptation to our problem of methods used to get estimates on the rate of approach to the equilibrium of Markov chains from the knowledge of the distribution of the return time of a single state (see, for instance, Ref. 18). The idea consists in embedding our Markov chain in another one with larger state space, where an auxiliary single state has been inserted.

Given M sufficiently large, we introduce two stochastic kernels $A_{s,\hat{s}}$ and $B_{\hat{s},s}$, where s ranges over $(\mathfrak{X}_{\leqslant N})^L$ endowed with the measure ρ and $\hat{s}$ over $(\mathfrak{X}_{\leqslant N})^L \cup \{e\}$ (where e is an auxiliary state) endowed with the measure $\hat{\rho}$, which is equal to ρ when restricted to $(\mathfrak{X}_{\leqslant N})^L$ and such that $\hat{\rho}(\{e\}) = 1$. The kernels $A_{s,\hat{s}}$ and $B_{\hat{s},s}$ are defined by

$$\begin{aligned}
A_{s,\hat{s}} &= P_{s,\hat{s}} - \Big(\inf_{t \in \mathscr{S}_M} P_{t,\hat{s}}\Big)\chi_{\mathscr{S}_M}(s)\chi_{\mathscr{S}_M}(\hat{s}), \qquad \hat{s} \in (\mathfrak{X}_{\leqslant N})^L \\
A_{s,e} &= \int \rho(ds')\Big(\inf_{t \in \mathscr{S}_M} P_{t,s'}\Big)\chi_{\mathscr{S}_M}(s) \\
B_{\hat{s},s} &= \delta_{\hat{s},s}, \qquad \hat{s} \in (\mathfrak{X}_{\leqslant N})^L \\
B_{e,s} &= \Big(\inf_{t \in \mathscr{S}_M} P_{t,s}\Big)\chi_{\mathscr{S}_M}(s) \Big/ \int \rho(ds')\Big(\inf_{t \in \mathscr{S}_M} P_{t,s'}\Big)
\end{aligned} \tag{2.12}$$

where $\delta_{\hat{s},s}$ is defined according to the standard convention:

$$\int \rho(ds)\, \delta_{\hat{s},s} f(s) = f(\hat{s}), \qquad \text{for any function } f$$

It is immediate to check that

$$P_{s,s'} = \int \hat{\rho}(d\hat{t}) A_{s,\hat{t}} B_{\hat{t},s'} \tag{2.13}$$

We define an nonhomogeneous Markov chain with time t ranging over $\frac{1}{2}\mathbb{Z}$ and state space $[(\mathfrak{X}_{\leqslant N})^L, \rho]$ for integral times and $[(\mathfrak{X}_{\leqslant N})^L \cup \{e\}, \hat{\rho}]$ for half-integral times. The transition probabilities from the time t to the time $t + \frac{1}{2}$ are given by the kernel $A_{s,\hat{s}}$ if t is an integer and by $B_{\hat{s},s}$ if t is a half-integer. It follows from (2.13) that the restriction of this Markov chain to integral times is defined by the kernel $P_{s,s'}$ with invariant measure $\tilde{v}(s)v(s)\rho(ds)$. We shall denote by P the stationary measure (with respect to integral shifts) on the space of sequences $(s_i)_{i \in (1/2)\mathbb{Z}}$. P induces on s_i with integral i the measure $\tilde{v}(s_i)v(s_i)\rho(ds)$.

Now we want to evaluate the probability of large return times to the state e. In order to get the factorization result given by Theorem 2.1, we shall make use of the following:

Proposition 2.4. There exist positive constants c_{18}, c_{19} such that for any $N > N_0$, $L > L_0$, $M > 1$ and for any $s, s' \in \mathscr{S}_M$

$$eP^n_{s,s'} / P^n_{s,s'} \leqslant \exp\big(c_{18}M^2 - ne^{-c_{19}M^2}\big) \tag{2.14}$$

where

$$eP^n_{s,s'} = P\big(s_{i+1/2} \neq e,\ i = 0, \ldots, n-1,\ s_n = s' \mid s_0 = s\big) \tag{2.15}$$

Proof. Let us assume that n is odd, $n = 2m + 1$ in order to fix the notation. For $s_1, \ldots, s_{2m} \in (\mathfrak{X}_{\leqslant N})^L$ we define

$$\psi(s_1, \ldots, s_{2m}) = \exp\left[-h(s_1)/2 - h(s_{2m})/2 \right] \prod_{i=1}^{2m-1} T_{s_i, s_{i+1}} \tag{2.16}$$

Given $1 \leqslant i_1 < \cdots < i_k \leqslant m - 1$ we have

$$P\big((s_{2i_l}, s_{2i_l+1}) \notin \mathscr{S}_M \times \mathscr{S}_M,\, l = 1, \ldots, k \,|\, s_0 = s, s_n = s'\big)$$

$$= \frac{\int \rho(ds_1) \ldots \rho(ds_{2m}) \exp\left[-W(s,s_1) - W(s_{2m},s') \right. \times \prod_{l=1}^k \left[1 - \chi_{\mathscr{S}_M}(s_{2i_l}) \chi_{\mathscr{S}_M}(s_{2i_l+1}) \right] \psi(s_1, \ldots, s_{2m})}{\int \rho(ds_1) \ldots \rho(ds_{2m}) \exp\left[-W(s,s_1) - W(s_{2m},s') \right] \psi(s_1, \ldots, s_{2m})}$$

$$= \frac{\int \rho(ds_1) \ldots \rho(ds_{2m}) \exp\left[-W(s,s_1) - W(s_{2m},s') \right] \times \prod_{l=1}^k \left[1 - \chi_{\mathscr{S}_M}(s_{2i_l}) \chi_{\rho_M}(s_{2i_l+1}) \right] \psi(s_1, \ldots, s_{2m})}{\int \rho(ds_1) \ldots \rho(ds_{2m}) \psi(s_1, \ldots, s_{2m})}$$

$$\times \frac{\int \rho(ds_1) \ldots \rho(ds_{2m}) \psi(s_1, \ldots, s_{2m})}{\int \rho(ds_1) \ldots \rho(ds_{2m}) \exp\left[-W(s,s_1) - W(s_{2m},s') \right] \psi(s_1, \ldots, s_{2m})}$$

$$\leqslant \left(2c_2 e^{-c_3 M^2}\right)^k c_5^2 e^{c_6 M^2} / \left(\tfrac{1}{2} e^{-2c_9 M^2}\right) \leqslant e^{c_{13} M^2 - k c_{14} M^2} \tag{2.17}$$

where we have assumed M large enough (this assumption can of course be removed in the final estimate) and we have used the fact that the probability measure proportional to $\psi(s_1, \ldots, s_{2m}) \rho(ds_1) \ldots \rho(ds_{2m})$ satisfies the superstability estimates like in the proof of (2.4) and (2.5) in Proposition 2.2.

Let us fix a constant α, $0 < \alpha < 1/2$. By using (2.17) we get for $s, s' \in \mathscr{S}_M$

$$P\big(\#\{ l \,|\, 1 \leqslant l \leqslant m-1, (s_{2i_l}, s_{2i_l+1}) \notin \mathscr{S}_M \times \mathscr{S}_M \} \geqslant \alpha n \,|\, s_0 = s, s_n = s'\big)$$

$$\leqslant e^{c_{13} M^2} \sum_{k=[\alpha n]}^{m} \binom{m}{k} e^{-k c_{14} M^2}$$

$$\leqslant e^{c_{13} M^2} e^{-([\alpha n]/2) c_{14} M^2} \left(1 + e^{-(1/2) c_{14} M^2}\right)^m$$

$$\leqslant e^{c_{15} M^2} e^{-n c_{16} M^2} \tag{2.18}$$

for M large enough.

Now we bound from below the probability

$$P(s_{i+1/2} = e \mid s_i = s, s_{i+1} = s')$$

with s and $s' \in \mathscr{S}_M$. We have for $s, s' \in \mathscr{S}_M$:

$$\begin{aligned} &P(s_{i+1/2} = e \mid s_i = s, s_{i+1} = s') \\ &\quad = A_{s,e}B_{e,s'}/P_{s,s'} = \inf_{t \in \mathscr{S}_M} P_{t,s'}/P_{s,s'} \\ &\quad = \frac{\inf_{t\in\mathscr{S}_M} \exp[-h(t)/2 - W(t,s') - h(s')/2] v(t)^{-1} v(s')}{\exp[-h(s)/2 - W(s,s') - h(s')/2] v(s)^{-1} v(s')} \\ &\quad \geqslant c_{10}^{-2} e^{-2(c_9 + c_{11})M^2} \end{aligned} \tag{2.19}$$

Let us define the events E, E^c by

$$E = \left\{ s_i, i \in \tfrac{1}{2}\mathbb{Z} : \#\left\{ 1 \leqslant l \leqslant \frac{n-3}{2} : (s_{2l}, s_{2l+1}) \notin \mathscr{S}_M \times \mathscr{S}_M \right\} \geqslant \frac{n}{4} \right\} \tag{2.20}$$

E^c is the complementary event of E.

We have for s and $s' \in \mathscr{S}_M$:

$$\begin{aligned} &eP^n_{s,s'}/P^n_{s,s'} \\ &\quad = P(s_{i+1/2} \neq e, i = 0, \ldots, n-1 \mid s_0 = s, s_n = s') \\ &\quad \leqslant P(E \mid s_0 = s, s_n = s') \\ &\qquad + P(s_{i+1/2} \neq e, i = 0, \ldots, n-1 \mid s_0 = s, s_n = s', E^c) \\ &\quad \leqslant e^{c_{15}M^2} e^{-nc_{16}M^2} + \left[1 - \inf_{t, t' \in \mathscr{S}_M} P(s_{i+1/2} = e \mid s_0 = t, s_n = t') \right]^{n/4 - 3/2} \\ &\quad \leqslant e^{c_{15}M^2} e^{-nc_{16}M^2} + \left(c_{10}^{-2} e^{-2(c_9 + c_{11})M^2} \right)^{n/4 - 3/2} \end{aligned} \tag{2.21}$$

where we used estimates (2.18) and (2.19). The result follows from (2.21).

Proof of Theorem 1. Let us introduce the following notation:

$$\begin{aligned} &\Pi^n_{ee} = P(s_{n+1/2} = e \mid s_{1/2} = e) \\ &e\Pi^n_{ee} = P(s_{i+1/2} \neq e, i = 1, \ldots, n+1, s_{n+1/2} = e \mid s_{1/2} = e) \qquad (2.22) \\ &\pi_e = P(s_{1/2} = e) = \left(\sum_{k=1}^{\infty} k_e \Pi^k_{ee} \right)^{-1} \end{aligned}$$

We have

$$e\Pi_{ee}^{n}=\int_{\mathscr{S}_M\times\mathscr{S}_M}\rho(ds)\,\rho(ds')\,B_{e,s}eP_{s,s'}^{n-1}A_{s',e}$$
$$=\int_{\mathscr{S}_M\times\mathscr{S}_M}\rho(ds)\,\rho(ds')\,B_{e,s}\big(eP_{s,s'}^{n-1}/P_{s,s'}^{n-1}\big)P_{s,s'}^{n-1}A_{s',e}$$
$$\leqslant\sup_{s,s'\in\mathscr{S}_M}\big(eP_{s,s'}^{n-1}/P_{s,s'}^{n-1}\big)\leqslant\exp\big[c_{18}M^2-(n-1)e^{-c_{19}M^2}\big]\qquad(2.23)$$

by Proposition 2.4.

On the other hand, by simple arguments similar to those used in the proofs of the above propositions we get

$$e\Pi_{ee}^{1}\geqslant\tfrac{3}{4}c_{10}^{-2}e^{-2c_{11}M^2}e^{-c_9M^2}\qquad(2.24)$$

so that by applying the result of Appendix B, we have for $N>N_0$, $L>L_0$ and M large enough

$$|\Pi_{ee}^{n}-\pi_e|\leqslant\exp\big(c_{20}M^2-ne^{-c_{21}M^2}\big)\qquad(2.25)$$

where c_{20} and c_{21} are suitable positive constants.

Let $s\in\mathscr{S}_M$. We have by the same argument used to prove (2.23):

$$P\big(s_{i+1/2}\neq e,\ i=1,\ldots,n-1,\ s_{n+1/2}=e\,|\,s_0=s\big)$$
$$\leqslant\exp\big(c_{18}M^2-ne^{-c_{19}M^2}\big)\qquad(2.26)$$

Therefore we get for $s\in\mathscr{S}_M$

$$(P^nA)_{s,e}-\pi_e=(eP^nA)_{s,e}+\sum_{r=0}^{n-1}(eP^rA)_{s,e}(\Pi_{ee}^{n-r}-\pi_e)-\pi_e\sum_{r=n}^{\infty}(eP^rA)_{s,e}$$
$$\leqslant\exp\big(c_{18}M^2-ne^{-c_{19}M^2}\big)+\sum_{r=0}^{n-1}\exp\big(c_{18}M^2-re^{-c_{19}M^2}\big)$$
$$\times\exp\big[c_{20}M^2-(n-r)e^{-c_{21}M^2}\big]$$
$$+\sum_{r=n}^{\infty}\exp\big(c_{18}M^2-re^{-c_{19}M^2}\big)$$
$$\leqslant\exp\big(c_{22}M^2-ne^{-c_{23}M^2}\big)\qquad(2.27)$$

where we have used the fact that

$$\sum_{r=0}^{\infty}(eP^rA)_{s,e}=1$$

which is true since the state e is recurrent.

For $s \in \mathscr{S}_M$ we have from Proposition 2.3 and Proposition 2.4

$$\begin{aligned}
&(B_e P^n)_{e,s}/v(s)\tilde{v}(s) \\
&\quad = P(s_{i+1/2} \neq e, i = 1, \ldots, n, s_{n+1} = s \mid s_{1/2} = e)/v(s)\tilde{v}(s) \\
&\quad = \int_{\mathscr{S}_M} \rho(ds')\, B_{e,s'} e P^n_{s',s}/v(s)\tilde{v}(s) \\
&\quad = \int_{\mathscr{S}_M} \rho(ds')\, B_{e,s'}(eP^n_{s',s}/P^n_{s',s})\left[P^n_{s',s}/v(s)\hat{v}(s)\right] \\
&\quad \leqslant \exp\left(c_{18}M^2 - ne^{-c_{19}M^2}\right)e^{c_{12}M^2}
\end{aligned} \tag{2.28}$$

We remark that

$$\pi_e \sum_{r=0}^{\infty} (B_e P^r)_{es} = v(s)\tilde{v}(s) \tag{2.29}$$

Indeed for any measurable set $F \subset (\mathfrak{X}_{\leqslant N})^L$:

$$\begin{aligned}
&\int_F \rho(ds)\, \pi_e \left[\sum_{r=0}^{\infty} (B_e P^r)_{e,s}\right] \\
&\qquad = P(s_0 \in F, s_{i+1/2} = e \text{ for some } i < 0) = P(s_0 \in F)
\end{aligned}$$

By the previous estimates we get for $s, s' \in \mathscr{S}_M$:

$$\begin{aligned}
&|T^n(s,s')/\lambda^n v(s)\tilde{v}(s') - 1| \\
&\quad = |P^n_{s,s'}/v(s')\tilde{v}(s') - 1| \\
&\quad = \Bigg| eP^n_{s,s'}/v(s')\tilde{v}(s') + \sum_{r=0}^{n-1} \left(B_e P^{n-r-1}\right)_{es'}\left[(P^r A)_{s,e} - \pi_e\right]/v(s')\hat{v}(s') \\
&\qquad - \pi_e \sum_{r=n}^{\infty} (B_e P^r)_{es'}/v(s')\tilde{v}(s') \Bigg| \\
&\quad \leqslant \exp\left(c_{18}M^2 - ne^{-c_{19}M^2}\right)e^{c_{12}M^2} \\
&\qquad + \sum_{r=0}^{n-1} \exp\left[c_{18}M^2 - (n-r-1)e^{-c_{19}M^2}\right]e^{c_{12}m^2}\exp\left(c_{22}M^2 - re^{-c_{23}M^2}\right) \\
&\qquad + \sum_{r=n}^{\infty} \exp\left(c_{18}M^2 - re^{-c_{19}M^2}\right)e^{c_{12}M^2} \\
&\quad \leqslant \exp\left(\bar{c}_1 M^2 - ne^{-\bar{c}_2 M^2}\right) \quad \blacksquare
\end{aligned} \tag{2.30}$$

3. SECTION 3

In this section we shall perform a polymer expansion of the complex partition function (1.23). This expansion will allow us to express the pressure as a convergent series of analytic functions and to prove then Theorem 1.1.

Using the definitions of Section 1 and following notation,

$$
\begin{aligned}
&W^{\omega}_{A_i,B_i,A_{i+1}}(\alpha_i, \beta_i, \alpha_{i+1}) \\
&\quad = H_{A_i\cup B_i\cup A_{i+1}}(\omega, \alpha_i \vee \beta_i \vee \alpha_{i+1}) - H^{<L}_{A_i\cup B_i\cup A_{i+1}}(\alpha_i \vee \beta_i \vee \alpha_{i+1}) \\
&\qquad - (1-\delta_{i,-p})\left[H_{A_i}(\omega,\alpha_i) - H^{<L}_{A_i}(\alpha_i)\right]/2 - (1-\delta_{i+1,p}) \\
&\qquad \times\left[H_{A_{i+1}}(\omega,\alpha_{i+1}) - H^{<L}_{A_{i+1}}(\alpha_{i+1})\right]/2 \qquad (3.1)\\
&W^{\omega}_{\Lambda,\Delta}(s_\Lambda, s_\Delta) = H_{\Lambda\cup\Delta}(\omega, s_\Lambda \vee s_\Delta) - H_\Lambda(\omega, s_\Lambda) - H_\Delta(\omega, s_\Delta) \\
&\qquad\qquad \text{for every} \quad \Lambda,\Delta \subset \mathbb{Z}, \quad \Lambda\cap\Delta = \varnothing
\end{aligned}
$$

(1.23) can be rewritten in the form

$$
\begin{aligned}
Z_{\Lambda_p,N}(\omega) = &\int \nu^N_{\Lambda_p}(ds_{\Lambda_p}) \exp\left[-h_L(\alpha_{-p})/2 - h_L(\alpha_p)/2\right] \\
&\times \prod_{i=-p}^{p-1} T_{N,L}(\alpha_i, \beta_i, \alpha_{i+1}) \prod_{i=-p}^{p-1} \exp\left[W^{\omega}_{A_i,B_i,A_{i+1}}(\alpha_i, \beta_i, \alpha_{i+1})\right] \\
&\times \prod_{i=-p}^{p-2} \prod_{j=i+2}^{p} \exp\left[W^{\omega}_{A_i,A_j}(\alpha_i, \alpha_j)\right] \\
&\times \prod_{i=-p}^{p} \prod_{j=-p}^{p-1} (1-\delta_{i,j})(1-\delta_{i-1,j}) \exp\left[W^{\omega}_{A_i,B_j}(\alpha_i, \beta_i)\right] \\
&\times \prod_{i=-p}^{p-2} \prod_{j=i+1}^{p-1} \exp\left[W^{\omega}_{B_i,B_j}(\beta_i, \beta_j)\right]
\end{aligned}
$$

Given p, we define $\mathscr{C}_p$ as the family of the sets C of the following types:

$$
\begin{aligned}
&C = \{A_i, A_j\}, \qquad -p \leqslant i,j \leqslant p, \qquad j \neq i-1, i, i+1 \\
&C = \{A_i, B_j\}, \qquad -p \leqslant i \leqslant p, \qquad -p \leqslant j \leqslant p-1, \qquad j \neq i, i-1 \\
&C = \{B_i, B_j\}, \qquad -p \leqslant i,j \leqslant p-1, \qquad j \neq i
\end{aligned}
$$

and $\mathscr{P}_p$ as the family of the sets C of the form

$$
C = \{A_i, B_i, A_{i+1}\}, \qquad \text{with} \quad -p \leqslant i \leqslant p-1
$$

For any $\Gamma \subset \mathscr{C}_p \cup \mathscr{P}_p$ we put $\mathscr{B}(\Gamma) = \{i : B_i \in \bigcup_{C\in\Gamma} C\}$ and $\mathscr{B}^c(\Gamma) =$

$\{-p, \ldots, p-1\}\backslash\mathscr{B}(\Gamma)$. The partition function can then be expressed as

$$
\begin{aligned}
Z_{\Lambda_p,N}(\omega) = \sum_{\Gamma C \mathscr{C}_p \cup \mathscr{P}_p} & \int \nu^N_{\Lambda_p \backslash \bigcup_{i\in\mathscr{B}^C(\Gamma)} B_i} \left(ds_{\Lambda_p \backslash \bigcup_{i\in\mathscr{B}^c(\Gamma)} B_i}\right) \\
& \times \exp\left[-h_L(\alpha_{-p})/2 - h_L(\alpha_p)/2\right] \\
& \times \prod_{i\in\mathscr{B}(\Gamma)} T_{N,L}(\alpha_i, \beta_i, \alpha_{i+1}) \prod_{C\in\Gamma} \varphi_C(s_C) \\
& \times \prod_{j\in\mathscr{B}^c(\Gamma)} T^{n+1}_{N,L}(\alpha_j, \alpha_{j+1})
\end{aligned}
\tag{3.2}
$$

where

$$
\varphi_c(s_c) \begin{cases} \exp\left[W^\omega_{A_i,A_j}(\alpha_i, \alpha_j)\right] - 1 & \text{if } C = \{A_i, A_j\} \\ \exp\left[W^\omega_{A_i,B_j}(\alpha_i, \beta_j)\right] + 1 & \text{if } C = \{A_i, B_j\} \\ \exp\left[W^\omega_{B_i,B_j}(\beta_i, \beta_j)\right] - 1 & \text{if } C = \{B_i, B_j\} \\ \exp\left[W^\omega_{A_i,B_i,A_{i+1}}(\alpha_i, \beta_i, \alpha_{i+1})\right] - 1, & \text{if } C = \{A_i, B_i, A_{i+1}\} \end{cases}
\tag{3.3}
$$

respectively.

Using the notation of Section 1 for the largest eigenvalue $\lambda_{N,L}$ and the corresponding left and right eigenvectors $\tilde{v}_{N,L}$ and $v_{N,L}$ of the transfer operator $T_{N,L}$, we can make a further expansion of the partition function:

$$
\begin{aligned}
Z_{\Lambda_p,N}(\omega) = \lambda^{[2p(n+1)+1]}_{N,L} \sum_{\Gamma\subset\mathscr{C}_p\cup\mathscr{C}_p} \sum_{I\subset\mathscr{B}^c(\Gamma)} \sum_{Y\subset I} & \int \nu^N_{\Lambda_p\backslash\bigcup_{i\in\mathscr{B}(\Gamma)} B_i} \left(ds_{\Lambda_p\backslash\bigcup_{i\in\mathscr{B}(\Gamma)} B_i}\right) \\
& \times \left\{\exp\left[-h_L(\alpha_{-p})/2 - h_L(\alpha_p)/2\right]/\lambda_{N,L}\right\} \\
& \times \prod_{i\in\mathscr{B}(\Gamma)} \left[T_{N,L}(\alpha_i, \beta_i, \alpha_{i+1})/\lambda^{n+1}_{N,L}\right] \\
& \times \prod_{C\in\Gamma} \varphi_C(s_C) \prod_{i\in\mathscr{B}^C(\Gamma)\backslash I} \Big\{\left[T^{n+1}_{N,L}(\alpha_i, \alpha_{i+1})/\lambda^{n+1}_{N,L}\right] \\
& \qquad\qquad \times\left[1 - \chi_M(\alpha_i)\chi_M(\alpha_{i+1})\right]\Big\} \\
& \times \prod_{i\in Y} \Big(\left\{T^{n+1}_{N,L}(\alpha_i, \alpha_{i+1})/\left[\lambda^{n+1}_{N,L} v_{N,L}(\alpha_i)\tilde{v}_{N,L}(\alpha_{i+1})\right] - 1\right\} \\
& \qquad \times \chi_M(\alpha_i)\chi_M(\alpha_{i+1})\Big) \\
& \times \prod_{i\in I} \left[v_{N,L}(\alpha_i)\tilde{v}_{N,L}(\alpha_{i+1})\chi_M(\alpha_i)\chi_M(\alpha_{i+1})\right]
\end{aligned}
\tag{3.4}
$$

where χ_M is the characteristic function of the set $\mathscr{S}^L_M$.

Now, following Refs. 11 and 7 and related references quoted in Ref. 7, we want to express the partition function (and the correlation functions) of our spin system in terms of those of a polymer gas, i.e., a gas composed of infinitely many species of molecules interacting only via hard core exclusion. Our goal will be then to show that the polymer gas is in the "small activity region" for an appropriate choice of the integers M, L, and n and for ω small enough. This will imply, by means of standard methods, that the free energy (and the correlation functions) can be expressed as a convergent series of local quantities and have the desired analyticity properties.

From now on, we shall often omit the indices L, N, in order to simplify the notation.

We denote by $\mathscr{G}_p$ the family of the sets $\{A_i, A_{i+1}\}$ with $-p \leqslant i \leqslant p-1$. If $C \in \mathscr{G}_p$, and $C = \{A_i, A_{i+1}\}$, we define

$$\begin{aligned} \varphi_c^1(s_c) &= 1 - \chi_M(\alpha_i)\chi_M(\alpha_{i+1}) \\ \varphi_c^2(s_c) &= \left\{ T_{N,L}^{n+1}(\alpha_i, \alpha_{i+1}) / \left[\lambda_{N,L}^{n+1} v(\alpha_i)\tilde{v}(\alpha_{i+1}) \right] - 1 \right\} \chi_M(\alpha_i)\chi_M(\alpha_{i+1}) \end{aligned} \tag{3.5}$$

Given $C \in \mathscr{C}_p \cup \mathscr{P}_p \cup \mathscr{G}_p$ we define the support of C, denoted by $\hat{C}$ as

$$\hat{C} = \bigcup_{A_i \subset C} A_i \bigcup_{B_i \subset C} B_i \bigcup_{i\,:\,B_i \text{ or } B_{i-1} \subset C} A_i$$

Let us consider a quadruple $R = (\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4)$ with $\Gamma_1 \subset \mathscr{C}_p$, $\Gamma_2 \subset \mathscr{P}_p$, $\Gamma_3 \cup \Gamma_4 \subset \mathscr{G}_p$. We call R admissible if the following conditions are satisfied:

(1) $\qquad \Gamma_3 \cap \Gamma_4 = \varnothing$

(2) $\qquad$ if $B_i \in \bigcup_{C \in \Gamma_1 \cup \Gamma_2} C$, then $\{A_i, A_{i+1}\} \notin \Gamma_3 \cup \Gamma_4$

Let now C_1 and C_2 belong to $\bigcup_{i=1}^4 \Gamma_i$. We say that C_1 and C_2 are connected if $\hat{C}_1 \cap \hat{C}_2 \neq \varnothing$. An admissible $R = (\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4)$ is called a polymer if for C and $C' \in \bigcup_{i=1}^4 \Gamma_i$ there exists a sequence C_j, $j = 1, \ldots, k$, such that $C_j \in \bigcup_{i=1}^4 \Gamma_i$, $C_1 = C$, $C_k = C'$, C_j and C_{j+1} are connected for $1 \leqslant j \leqslant k-1$.

Let $R = (\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4)$ be a polymer and let $I(R)$ be the set of those indices i such that either $B_i \in \bigcup_{C \in \Gamma_1 \cup \Gamma_2} C$ or $\{A_i, A_{i+1}\} \in \Gamma_3$ and $J(R) = \{i : A_i \in \bigcup_{C \in \Gamma_1 \cup \Gamma_4} A_i\}$. We define the support of R by

$$\hat{R} = \bigcup_{i \in I(R)} (A_i \cup B_i \cup A_{i+1}) \bigcup_{j \in J(R)} A_j \tag{3.6}$$

For every polymer R the support $\hat{R}$ can then be decomposed as a union of disjoint intervals:

$$\begin{aligned} \hat{R} = \bigcup_{i=1}^{k} G_i, \qquad &\text{where} \quad G_i = A_{l_i} \quad \text{or} \\ G_i = A_{l_i} \cup B_{l_i} \cup A_{l_i+1} \cup \cdots \cup B_{l_i+m_i} \cup A_{l_i+m_i+1}, \qquad &m_i \geqslant 0 \end{aligned} \tag{3.7}$$

We define a measure μ_R^N on the configurations in $\hat{R}$ as $\mu_R^N(ds_{\hat{R}}) = \prod_{i=1}^k \mu_{G_i}^N(ds_{G_i})$, where μ_G^N denotes the probability measure on the configurations in G absolutely continuous with respect to ν_G^N, with density given by

$$\frac{d\mu_G^N}{d\nu_G^N} = \frac{\tilde{u}_l(\alpha_l)u_l(\alpha_l)}{\mathcal{N}_G}, \qquad \text{if} \quad G = A_l$$

$$\frac{d\mu_G^N}{d\nu_G^N} = \frac{\tilde{u}_l(\alpha_l)T(\alpha_l, \beta_l, \alpha_{l+1}) \cdots T(\alpha_{l+m}, \beta_{l+m}, \alpha_{l+m+1})u_{l+m=1}(\alpha_{l+m+1})}{(\lambda^{m(n+1)}\mathcal{N}_G)} \tag{3.8}$$

$$\text{if} \quad G = A_l \cup B_l \cup A_{l+1} \cup \cdots \cup B_{l+m} \cup A_{l+m+1}$$

where $\tilde{u}_l = \tilde{v}\chi_M$, $-p+1 \leqslant l \leqslant p$, $u_l = v\chi_M$, $-p \leqslant l \leqslant p-1$ $\tilde{u}_{-p} = u_p = \exp(-h_L/2)/\lambda^{1/2}$, and $\mathcal{N}_G$ is the normalization. We put $\mathcal{N}_R = \prod_{i=1}^k \mathcal{N}_{G_i}$, if $\hat{R} = \bigcup_{i=1}^k G_i$.

Due to P.3 of Section 1, the correlation functions of the measures μ_R^N satisfy superstability estimates:

$$\int \nu_{\hat{R}\setminus\Delta}^N(ds_{\hat{R}\setminus\Delta}) \frac{d\mu_R^N}{d\nu_{\hat{R}}^N}(ds_{\hat{R}}) \leqslant \exp\left[-\sum_{x\in\Delta}(A^*|s_x|^2 - \delta^*)\right] \tag{3.9}$$

for every $L > L_0$, $N > N_0$, $R, \Delta \subset \hat{R}$.

We associate to a polymer R its complex valued activity $\zeta^N(R)$ defined as

$$\zeta^N(R) = \int \mu_R^N(ds_{\hat{R}}) \prod_{C\in\Gamma_1\cup\Gamma_2} \varphi_c(s_c) \prod_{C\in\Gamma_3} \varphi_c^1(s_c) \sum_{C\in\Gamma_4} \varphi_c^2(s_c) \tag{3.10}$$

One can easily verify that the partition function can be expressed in the following way:

$$Z_{\Lambda_p,N}(\omega) = \lambda^{2p(n+1)+1}\left[v\exp(-h/2), \chi_M\lambda^{-1/2}\right]^2 (v\tilde{v}, \chi_M)^{2p-1}$$

$$\times\left[1 + \sum_{n\geqslant 1} \sum_{\substack{R_1,\ldots,R_n \\ \hat{R}_i\subset\Lambda_p,\ \hat{R}_i\cap\hat{R}_j=\emptyset\ i\neq j}} \prod_{i=1}^n \bar{\zeta}^N(R_i)\right] \tag{3.11}$$

where $(\cdot,\cdot)$ denotes the scalar product in $L^2[(\mathfrak{X}_{\leqslant N})^L, \nu_L^N]$ and

$$\bar{\zeta}^N(R) = \zeta^N(R)\mathcal{N}_R / \left\{(v\tilde{v}, \chi_M)^{\#\{i\,:\, i\in\{-p,p\}, A_i\subset\hat{R}\}}\right.$$

$$\left.\times\left[v\exp(-h/2), \chi_M\lambda^{-1/2}\right]^{\#\{i\,:\,A_i\subset\hat{R}\}\cap\{-p,p\}}\right\} \tag{3.12}$$

The term within square brackets in Eq. (3.11) can be interpreted as the partition function of a polymer system with hard core interaction and activity $\bar{\zeta}^N$. Note that $\bar{\zeta}^N$ is, in general, no translationally invariant. However, $\bar{\zeta}^N(R) = \bar{\zeta}^N(R')$ if R' is obtained from R by a translation of a multiple of $(n+1)L$ and both R and R' do not intersect $A_{-p} \cup A_p$.

In the sequel we shall estimate $\zeta^N(R)$ and, consequently, $\bar{\zeta}^N(R)$ uniformly in N and obtain a uniform estimate of the complex free energy by means of a cluster expansion. The result is contained in the following lemma:

Lemma 3.1. For any given n, M large enough, we can choose $L_1(n,M)$, $\omega_1(L,n,M)$ such that if $L \geqslant L_1(n,M)$, $\omega \leqslant \omega_1(L,n,M)$ for every polymer $R(\Gamma_1,\Gamma_2,\Gamma_3,\Gamma_4)$, every $N \geqslant N_0$:

$$|\zeta^N(R)| \leqslant \prod_{C \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4} \hat{g}_c(M,n,L,\omega) \tag{3.13}$$

where, for $C \in \mathscr{C}_p$

$$\hat{g}_c(M,n,L,\omega) = \max\Big\{ 12M^2n^2(1+\omega d)\ln(nL+1)/\big[r_c\tilde{F}(r=L)\big],$$
$$6\bar{C}\exp\Big[-\frac{A^*}{L8}M^2\ln(r_c+1)\Big]\Big\}$$

where r_C is the number of blocks of the type A and B between V and V' if $C = \{V, V'\}$ and $\bar{C}$ is a fixed constant, whereas

$$\hat{g}_c(M,n,L,\omega)$$
$$= \max\Bigg\{ 100M^2\ln(nL+1)n\bigg[\frac{1}{\ln(L+1)\tilde{F}(L)} + \omega L\bar{A}\bigg] + 100nL\omega\bar{\delta},$$
$$6\bar{C}\exp\Big(-\frac{A^*}{48}M^2\Big), \exp\Big[\bar{C}_1M^2 - n\exp\big(-\bar{C}_2M^2\big)\Big]\Bigg\} \tag{3.14}$$

for $C \in \mathscr{P}_p \cup \mathscr{G}_p$.

$\bar{C}_1, \bar{C}_2$ are defined in Section 2. $\tilde{F}(r) = \min[F(r), \bar{F}(r)]$.

Proof. We start from the expression (3.10) and, first of all, we estimate the factors φ_C^2, $C \in \Gamma_4$, using Theorem 2.1:

$$|\varphi^2_{\{A_i,A_{i+1}\}}(\alpha_i,\alpha_{i+1})| \leqslant \exp\big(\bar{C}_1M^2 - ne^{-\bar{C}_2M^2}\big) \tag{3.15}$$

From (3.15) we get

$$|\zeta^N(R)| \leqslant \exp\Big[(\#\Gamma_4)\big(\bar{C}_1M^2 - ne^{-\bar{C}_2M^2}\big)\Big]$$
$$\times \int \mu_R^N(ds_{\hat{R}}) \prod_{C \in \Gamma_1 \cup \Gamma_2} |\varphi_C(s_C)| \prod_{u \in \Gamma_3} |\varphi_u^1(s_u)| \tag{3.16}$$

Now we bound the moduli of the factors appearing in (3.16) in terms of the values of the spins of the corresponding blocks.

If $C \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$

$$|\varphi_C(s_C)| \leqslant \sum_{V \in C} \bar{\varphi}_C(s_V) \tag{3.17}$$

where

$$\ln\left[1 + \bar{\varphi}_C(s_V)\right]$$
$$= \sum_{x \in V} |s_x|^2 nL(1 + \omega d)/d(V, V')^2 \ln\left[d(V, V') + 1\right] F\left[d(V, V')\right]$$

for $C \in \Gamma_1$, $C = \{V, V'\}$

$$\ln\left[1 + \bar{\varphi}_C(s_V)\right] = 4 \sum_{x \in V} |s_x|^2 \left\{\left[(L-1)\ln(L+1)F(L)\right]^{-1} + 2\omega\bar{A}\right\} + 2\omega nL\bar{\delta}$$

for $C \in \Gamma_2$ where $d(V, V')$ is the distance between the blocks V and V', $\tilde{F}(r) = \min[F(r), \bar{F}(r)]$ [see (1.3) and (1.8)].

We have used $|e^r - 1| \leqslant e^{|r|} - 1$, $\forall r \in \mathbb{C}$, $|xy| \leqslant (x^2 + y^2)/2$, $\forall x, y \in \mathbb{R}$, $|e^{x+y} - 1| \leqslant |e^{2x} - 1| + |e^{2y} - 1|$, $x, y \geqslant 0$.

For $C \in \Gamma_3$, $\bar{\varphi}_C(s_V) = 1 - \chi_M(s_V)$.

Now we insert the estimates given in (3.17) in the integral on the right-hand side of (3.16) and expand the products, getting a sum of factorized terms. We call $\mathscr{D}_R$ the set of the maps $D: \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \to (\bigcup_{i=-p}^{p} A_i) \cup (\bigcup_{i=-p}^{p} B_i)$ such that $D(C) \in C$. We get

$$\int \mu_R^N(dS_{\hat{R}}) \prod_{C \in \Gamma_1 \cup \Gamma_2} |\varphi_C(s_C)| \prod_{C \in \Gamma_3} |\varphi_C^1(s_C)|$$
$$\leqslant \sum_{D \in \mathscr{D}_R} \int \mu_R^N(ds_{\hat{R}}) \prod_{C \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3} \bar{\varphi}_C(s_{D(C)}) \equiv \sum_{D \in \mathscr{D}_R} \mathscr{T}_{R,D} \tag{3.18}$$

Now we want to estimate a single term $\mathscr{T}_{R,D}$ appearing on the right-hand side of (3.18). We define a function r on $\mathscr{C}_p \cup \mathscr{P}_p \cup \mathscr{G}_p$ by $r_C = 1$ for $C \in \mathscr{P}_p \cup \mathscr{G}_p$, $r_C = \sharp\{$blocks of type A and B lying between V and $V'\}$ if $C = \{V, V'\} \in \mathscr{C}_p$.

Given a polymer $R = (\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4)$, $V \in \bigcup_{C \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3} C$ and $D \in \mathscr{D}_R$, let $r_1^V \ldots r_{k_V}^V$ be the district values, in increasing order, assumed by the function r on $D^{-1}(V)$ [we shall put $k_V = 0$ if $D^{-1}(V) = \emptyset$]. Notice that $k_V \leqslant \sharp(D^{-1}(V))$.

Let $M(r) = M[r\ln(r+1)]^{1/2}$ and, for $x \in V$, $l(x) = \{\ln[d(x, \mathbb{Z}\setminus V) + 1]\}^{1/2}$. We set, for $0 < j \leqslant k_V$:

$$\chi_{V,j}(s_V) = \begin{cases} 1, & \text{if } |s_x| \leqslant M(r_j^V)l(x) \\ 0, & \text{otherwise} \end{cases} \tag{3.19}$$

and

$$\chi_{V,0}(s_V) = 0, \qquad \chi_{V,k_V+1}(s_V) = 1$$

Finally,

$$\bar{\chi}_{V,j}(s_V) = \chi_{V,j}(s_V) - \chi_{V,j-1}(s_V), \qquad 1 \leqslant j \leqslant k_V + 1 \tag{3.20}$$

We have $\sum_{j=1}^{k_V+1} \bar{\chi}_{V,j} = 1$.

Now we denote by $\mathscr{F}_{R,D}$ the set of the maps $F: \bigcup_{C \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3} C \to \mathbb{Z}_+$ such that $1 \leqslant F(V) \leqslant k_V + 1$. We have

$$\begin{aligned}
\mathscr{T}_{R,D} &= \int \mu_R^N(ds_{\hat{R}}) \prod_{C \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3} \bar{\varphi}_C(s_{D(C)}) \\
&= \int \mu_R^N(ds_{\hat{R}}) \prod_{V \in \bigcup_{C \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3} C} \; \prod_{C \in D^{-1}(V)} \left[\bar{\varphi}_C(s_V) \sum_{j=1}^{k_V+1} \bar{\chi}_{V,j}(s_V) \right] \\
&= \sum_{F \in \mathscr{F}_{R,D}} \int \mu_R^N(ds_{\hat{R}}) \prod_{V \in \bigcup_{C \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3} C} \left[\bar{\chi}_{V,F(V)}(s_V) \prod_{C \in D^{-1}(V)} \bar{\varphi}_C(s_V) \right] \\
&\equiv \sum_{F \in \mathscr{F}_{R,D}} \mathscr{T}_{R,D,F}
\end{aligned} \tag{3.21}$$

We shall estimate the quantities $\mathscr{T}_{R,D,F}$ in the following way. The contributions coming from "large-distance bonds" $C \in \mathscr{C}_p$ will be estimated with the upper bound on the moduli of the spins, implied by the presence of the functions $\bar{\chi}$'s, if this upper bound is sufficiently small. The contributions to the integral of the remaining factors will be evaluated by using the lower bounds on the moduli of the spins also implicit in the functions $\bar{\chi}$'s and the fact that high values of the spins are depressed by the superstable measure μ_R^N.

Thus we separate into two parts the product $\prod_{C \in D^{-1}(V)} \bar{\varphi}_C(s_V)$ which appears in (3.21) and, using $\bar{\chi}_{V,F(V)} = \chi_{V,F(V)}(1 - \chi_{V,F(V)-1})$, write

$$\begin{aligned}
&\bar{\chi}_{V,F(V)}(s_V) \prod_{C \in D^{-1}(V)} \bar{\varphi}_C(s_V) \\
&\quad = \left\{ \chi_{V,F(V)}(s_V) \prod_{\substack{C \in D^{-1}(V): \\ r_c \geqslant r_{F(V)}^V}} \bar{\varphi}_C(s_V) \right\} \\
&\qquad \times \left\{ \left[1 - \chi_{V,F(V)-1}(s_V)\right] \prod_{\substack{C \in D^{-1}(V): \\ r_c < r_{F(V)}^V}} \bar{\varphi}_C(s_V) \right\}
\end{aligned} \tag{3.22}$$

Let us consider the first of the two factors appearing on the right-hand side of (3.22). If in this factor a $C \in D^{-1}(V) \cap \mathscr{G}_p$ appears, then necessarily $r^V_{F(V)} = r_C = 1$ and $\chi_{V,F(V)}(s_V)\bar{\varphi}_C(s_V) = \chi_M(s_V)[1 - \chi_M(s_V)] = 0$. In the case there is no such C we have

$$\chi_{V,F(V)}(s_V) \prod_{\substack{C \in D^{-1}(V) \\ r_C \geqslant r^V_{F(V)}}} \bar{\varphi}_C(s_V)$$

$$\leqslant \prod_{\substack{C \in D^{-1}(V) \cap \mathscr{C}_p \\ r_C \geqslant r^V_{F(V)}}} \left| \exp\left[M^2 r_C \ln(r_C + 1) \ln(nL + 1) n^2 L^2 \times \frac{1 + \omega d}{(r_C L)^2 \ln(r_C L + 1) \tilde{F}(r_C L)} \right] - 1 \right|$$

$$\times \prod_{\substack{C \in D^{-1}(V) \cap \mathscr{C}_p \\ r_C \geqslant r^V_{F(V)}}} \left| \exp\left\{ 4M^2 \ln(nL + 1) nL \left[\frac{1}{(L-1)\ln(L+1)\tilde{F}(L)} + 2\omega\bar{A} \right] + 8\omega n L \bar{\delta} \right\} - 1 \right| \tag{3.23}$$

For every M, n, we can choose L so large and ω so small that we can bound the right-hand side of (3.23) by using the inequality $|e^{|x|} - 1| < 2|x|$ which is certainly true for $|x| < 1$ and get

$$\chi_{V,F(V)}(s_V) \prod_{\substack{C \in D^{-1}(V) \\ r_C \geqslant r^V_{F(V)}}} \bar{\varphi}_C(s_V)$$

$$\leqslant \prod_{\substack{C \in D^{-1}(V) \cap \mathscr{C}_p \\ r_C \geqslant r^V_{F(V)}}} \left[2M^2 n^2 (1 + \omega d) \ln(nL + 1) / r_C \tilde{F}(r_C L) \right]$$

$$\times \prod_{\substack{C \in D^{-1}(V) \cap \mathscr{C}_p \\ r_C \geqslant r^V_{F(V)}}} \left\{ 16M^2 \ln(nL + 1) n \left[\frac{1}{\ln(L+1)\tilde{F}(L)} + \omega L \bar{A} \right] + 16\omega n L \bar{\delta} \right\} \tag{3.24}$$

Now we consider the second factor on the right-hand side of (3.22). We have

$$\prod_{\substack{C\in D^{-1}(V)\\ r_C\leqslant r^V_{F(V)}}} \bar{\varphi}_C(s_V)$$

$$\leqslant \prod_{\substack{C\in D^{-1}(V)\cap(\mathscr{C}_p\cup\mathscr{C}_p)\\ r_C< r^V_{F(V)}}} \bar{\varphi}_C(s_V)$$

$$\leqslant \exp\left\{\sum_{x\in V}|s_x|^2 nL(1+\omega d)\sum_{r=1}^{\infty}\left[(rL)^2\ln(rL+1)F(rl)\right]^{-1}\right\}$$

$$\times\exp\left(16\sum_{x\in V}|s_x|^2\left((L\ln(L+1)F(L))^{-1}+\omega\bar{A}\right)+16\omega nL\bar{\delta}\right)$$

$$\equiv\exp\left(\hat{\gamma}(n,\omega,L)\sum_{x\in V}|s_x|^2+16\omega nL\bar{\delta}\right) \tag{3.25}$$

where

$$\hat{\gamma}(n,\omega,L)=n(1+\omega d)\sum_{r=1}^{\infty}\left[r^2L\ln(rL+1)F(rl)\right]^{-1}$$
$$+16\left[\frac{1}{L\ln(L+1)F(L)}+\omega\bar{A}\right] \tag{3.26}$$

We note that, given n, $\hat{\gamma}(n,\omega,L)\leqslant O([L\ln LF(L)]^{-1})+O(|\omega|)$.

In the following formulas we shall omit the dependence of $\hat{\gamma}$ on n, ω, and L and we shall assume that n, L, and ω are chosen in such a way that $\hat{\gamma}<A^*/3$.

We put $\hat{C}=\int_{\bar{x}}\nu(ds)\exp(-A^*s^2/3+\delta^*)$ and, for every real M, we shall denote by $\hat{\chi}_M$ the characteristic function of the interval $[0,M]$, $\chi_M^C=1-\hat{\chi}_M$.

Using (3.25) and superstability estimates we get

$$\int\mu_R^N(ds_{\hat{R}})\prod_{\substack{V\in\bigcup_{C\in\Gamma_1\cup\Gamma_2\cup\Gamma_3}C\\ D^{-1}(V)\neq\varnothing}}\left[1-\chi_{V,F(V)-1}(s_V)\right]\prod_{\substack{C\in D^{-1}(V)\\ r_C<r^V_{F(V)}}}\bar{\varphi}_C(s_V)$$

$$\leqslant\int\mu_R^N(ds_{\hat{R}})\prod_{\substack{V\in\bigcup_{C\in\Gamma_1\cup\Gamma_2\cup\Gamma_3}C\\ D^{-1}(V)\cap\{C\,:\,r_C<r^V_{F(V)}\}=\varnothing}}\left(e^{16\nu nL\bar{\delta}}\sum_{x\in V}\hat{\chi}^C_{M(r^V_{F(V)-1})l(x)}(|s_x|)e^{\hat{\gamma}|s_x|^2}\right)$$

$$\times\sum\sum_{\Delta\subset V\setminus\{x\}}\left(\prod_{y\in\Delta}\hat{\chi}_M(|s_y|)e^{\hat{\gamma}|s_y|^2}\prod_{z\in V\setminus(\Delta\cup\{x\})}\hat{\chi}^C_M(|s_z|)e^{\hat{\gamma}|s_z|^2}\right)$$

$$\leqslant \prod_{\substack{V\in\bigcup_{C\in\Gamma_1\cup\Gamma_2\cup\Gamma_3}C \\ D^{-1}(V)\cap\{C\,:\,r_C<r^V_{F(V)}\}\neq\varnothing}} e^{16\omega nL\bar{\delta}}\left(\sum_{x\in V}\int_{|s|>M(r^V_{F(V)-1})l(x)} \nu(ds)\, e^{(\hat{\gamma}-A^*)|s|^2+\delta^*}\right)$$

$$\times \sum_{\Delta\subset V\setminus\{x\}}\left(e^{\hat{\gamma}M^2|\Delta|}\left(\int_{|s|>M}\nu(ds)\,e^{(\hat{\gamma}-A^*)|s|^2+\delta^*}\right)^{|V|-|\Delta|-1}\right)$$

$$\leqslant \prod_{\substack{V\in\bigcup_{C\in\Gamma_1\cup\Gamma_2\cup\Gamma_3}C \\ D^{-1}(V)\cap\{C\,:\,r_C<r^V_{F(V)}\}\neq\varnothing}} e^{16\omega nL\bar{\delta}}$$

$$\times\left\{\sum_{x\in V}\int_{|s|>M(r^V_{F(V)-1})l(x)}\nu(ds)\,e^{-(2/3)A^*|s|^2+\delta^*}\right.$$

$$\left.\times\left[e^{\hat{\gamma}M^2}+\int_{|s|>M}\nu(ds)\,e^{-(2/3)A^*|s|^2+\delta^*}\right]^{|V|}\right\}$$

$$\leqslant \prod_{\substack{V\in\bigcup_{C\in\Gamma_1\cup\Gamma_2\cup\Gamma_3}C \\ D^{-1}(V)\cap\{C\,:\,r_C<r^V_{F(V)}\}\neq\varnothing}} e^{16\omega nL\bar{\delta}}\hat{C}\sum_{x\in V}\exp\left[-\frac{A^*}{3}M^2\left(r^V_{F(V)-1}\right)l^2(x)\right]$$

$$\times\left(e^{\hat{\gamma}M^2}+\hat{C}e^{-(A^*/3)M^2}\right)^{nL} \tag{3.27}$$

It follows that

$$\int \mu^N_R(ds_{\hat{R}}) \prod_{\substack{V\in\bigcup_{C\in\Gamma_1\cup\Gamma_2\cup\Gamma_3}C \\ D^{-1}(V)\neq\varnothing}}\left[1-\chi_{V,F(V)-1}(s_V)\right]\prod_{\substack{C\in D^{-1}(V) \\ r_C<r^V_{F(V)}}}\bar{\varphi}_C(s_V)$$

$$\leqslant \prod_{\substack{V\in\bigcup_{C\in\Gamma_1\cup\Gamma_2\cup\Gamma_3}C \\ D^{-1}(V)\cap\{C\,:\,r_C<r^V_{F(V)}\}\neq\varnothing}}\left\{\exp(16\omega nL\bar{\delta})\hat{C}\exp\left[-(A^*/12)M^2\left(r^V_{F(V)-1}\right)\right]\right.$$

$$\left.\times\left(\sum_{r=1}^{\infty}r^{-(A^*/6)M^2}\right)\left(e^{(\hat{\gamma}M^2)}+\hat{C}e^{-(A^*/3)M^2}\right)^{nL}\right\}$$

$$\leqslant \prod_{\substack{V\in\bigcup_{C\in\Gamma_1\cup\Gamma_2\cup\Gamma_3}C \\ D^{-1}(V)\cap\{C\,:\,r_C<r^V_{F(V)}\}\neq\varnothing}} C'(M,n,L,\omega)\exp\left[-(A^*/12)M^2\left(r^V_{F(V)-1}\right)\right] \tag{3.28}$$

where it is clear that for every M and n, we can find L so large and ω so small that $C'(M,n,L,\omega)$ is less than some fixed constant $\bar{C}$.

We observe now that for every block V there are at most four elements C's of $\mathscr{C}_p \cup \mathscr{P}_p \cup \mathscr{G}_p$ such that $V \in C$ and $r_C = 1$, whereas for $r > 1$ there are at most two C's belonging to $\mathscr{C}_p \cup \mathscr{C}_p \cup \mathscr{G}_p$ such that $V \in C$ and $r_C = r$ (in fact in this case they belong to $\mathscr{C}_p$). Then, if $\tilde{M}(r) = M(\ln(r+1))^{1/2}$,

$$\frac{1}{4} \sum_{C \in D^{-1}(V) \cap \{C \,:\, r_C < r^V_{F(V)}\}} \tilde{M}^2(r_C) = \frac{1}{4} \sum_{C \in D^{-1}(V) \cap \{C \,:\, r_C \leqslant r^V_{F(V)-1}\}} \tilde{M}(r_C)$$

$$\leqslant M^2\big(r^V_{F(V)-1}\big)$$

It follows for $L, \omega : C'(M, n, L, \omega) < \bar{C}$:

$$\int \mu_R^N(ds_{\hat{R}}) \prod_{\substack{V \in \bigcup_{C \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3} C \\ D^{-1}(V) \neq \varnothing}} \big[1 - \chi_{V, F(V)-1}(s_V)\big] \prod_{\substack{C \in D^{-1}(V) \\ r_C < r^V_{F(V)}}} \bar{\varphi}_C(s_V)$$

$$\leqslant \prod_{\substack{V \in \bigcup_{C \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3} C \\ D^{-1}(V) \neq \varnothing}} \prod_{\substack{C \in \mathscr{C}_p \cup \mathscr{C}_p \cup \mathscr{G}_p \\ C \ni V,\, r_C \leqslant r^V_{F(V)-1}}} \bar{C} \exp\left[-\frac{A^*}{48} \tilde{M}^2(r_C)\right] \tag{3.29}$$

By inserting in the decomposition (3.22) the estimates (3.24) and (3.29) we get for every $D \in \mathscr{D}_R$ and every $F \in \mathscr{F}_{R,D}$:

$$\mathscr{F}_{R,D,F} = \int \mu_R^N(ds_{\hat{R}}) \prod_{V \in \bigcup_{C \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3} C} \bar{\chi}_{V, F(V)}(s_V) \prod_{\grave{C} \in D^{-1}(V)} \bar{\varphi}_C(s_V)$$

$$\leqslant \prod_{C \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3} \tilde{g}_C \tag{3.30}$$

where if $C \in \mathscr{C}_p \cup \mathscr{G}_p$:

$$\tilde{g}_C = \max\left\{16 M^2 \ln(nL+1) n \left[\frac{1}{\ln(L+1)\tilde{F}(L)} + \omega L \bar{A}\right] + 16 n L \omega \bar{\delta},\right.$$

$$\left.\bar{C} e^{-(A^*/48) M^2}\right\} \tag{3.31}$$

and, if $C \in \mathscr{C}_p$;

$$\tilde{g}_C = \max\left\{2M^2 n^2 (1 + \omega d) \ln(nL+1) / r_C \tilde{F}(r_C L),\right.$$

$$\left.\bar{C} \exp\left[-\frac{A^*}{48} M^2 \ln(r_C + 1)\right]\right\}$$

We can now bound $\mathscr{T}_R$, using combinatorial estimates of the cardinalities of $\mathscr{D}_R$ and $\mathscr{F}_{R,D}$. We have

$$\begin{aligned}\mathscr{T}_R &= \sum_{D\in\mathscr{D}_R}\sum_{F\in\mathscr{F}_{R,D}}\mathscr{T}_{R,D,F}\\ &\leqslant \sum_{D\in\mathscr{D}_R}\prod_{V\in\bigcup_{C\in\Gamma_1\cup\Gamma_2\cup\Gamma_3}C}\left[k_V(D)+1\right]\prod_{C\in\Gamma_1\cup\Gamma_2\cup\Gamma_3}\tilde{g}_C\\ &\leqslant \sum_{D\in\mathscr{D}_R}\prod_{V\in\bigcup_{C\in\Gamma_1\cup\Gamma_2\cup\Gamma_3}C}\left[k_V(D)+1\right]\left(\frac{1}{2}\right)^{k_V(D)}\prod_{C\in\Gamma_1\cup\Gamma_2\cup\Gamma_3}(2\tilde{g}_C)\end{aligned}$$

The last inequality comes from the fact that $\forall D\in\mathscr{D}_R$

$$\sum_{V\in\bigcup_{C\in\Gamma_1\cup\Gamma_2\cup\Gamma_3}C}k_V(D)\leqslant \#(\Gamma_1\cup\Gamma_2\cup\Gamma_3)$$

Since $(k+1)(\frac{1}{2})^k\leqslant 1$, $\forall k\geqslant 0$ and $\#\mathscr{D}_R\leqslant 3^{\#(\Gamma_1\cup\Gamma_2\cup\Gamma_3)}$ we have

$$\mathscr{T}_R\leqslant\prod_{C\in\Gamma_1\cup\Gamma_2\cup\Gamma_3}6\tilde{g}_C \tag{3.32}$$

and the thesis of the lemma follows from (3.16), (3.32).

We can now easily obtain estimates on $\bar{\zeta}^N$ from those on ζ^N given in Lemma 2.1. Choosing $\overline{M}$ such that (2.7), (A8) hold we have for every $N>N_0$, $L>L_0$, $M>\overline{M}$

$$\begin{gathered}\tfrac{3}{4}\leqslant(v\tilde{v},\chi_{\overline{M}})\leqslant(v\tilde{v},\chi_M)\leqslant 1\\ \tfrac{3}{4}\mathscr{N}^{-1}\leqslant\left[v\exp(h/2),\chi_M\lambda^{-1/2}\right]\leqslant\mathscr{N}\\ \mathscr{N}_R\leqslant\mathscr{N}^2\end{gathered}\tag{3.33}$$

For every polymer R we have that

$$|R|\leqslant 3^{\#(\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4)} \tag{3.34}$$

where $|R|$ denotes the number of blocks of type A or B contained in $\hat{R}$. Using (3.33), (3.36) and Lemma 3.1 we obtain immediately for every polymer $R=(\Gamma_1,\Gamma_2,\Gamma_3,\Gamma_4)$, $1/2\leqslant\sigma<1$ any given n, $N\geqslant N_0$, $M\geqslant\overline{M}$:

$$|\bar{\zeta}(R)|\leqslant\sigma^{|R|}\prod_{C\in\Gamma_1\cup\Gamma^2\cup\Gamma_3\cup\Gamma_4}g_C \tag{3.35}$$

where $g_C=2^3\mathscr{N}^4\hat{g}_C$, if we choose L,ω^{-1} big enough.

We have obtained estimates on the activities of the polymers which are independent of N. This allows us to remove the cutoff N.

Let, for $L>L_0$, $\lambda_L=\lim_{N\to\infty}\lambda_{N,L}$ (see p. 2 of Section 1). Without worrying about the uniqueness of the limits, which would be, however, not difficult to obtain, we can find a subsequence N_i such that

$$\begin{aligned}\tfrac{3}{4} &\leqslant \lim_{i\to\infty}(v_{L,N_i}\tilde{v}_{L,N_i}, \chi_M) = (v\tilde{v}, \chi_M) \leqslant 1\\ \tfrac{3}{4}\mathscr{N}^{-1} &\leqslant \lim_{i\to\infty}\left[v_{L,N_i}\exp(-hL/2), \chi_M\lambda_{N_i,L}^{-1/2}\right]\\ &\leqslant \mathscr{N} = \left[v\exp(-h_L/2), \chi_M\lambda_L^{-1/2}\right]\end{aligned} \tag{3.36}$$

where the notation $(v\tilde{v}, \chi_M)$ and $[v\exp(-h_L/2), \chi_M\lambda_L^{-1/2}]$ is just formal and is used to remind us where these terms come from. Similarly we can assume that for the same subsequence N_i,

$$\lim_{i\to\infty}\bar{\zeta}^{N_i}(R) = \bar{\zeta}(R)$$

for every Λ_p and for every polymer R in Λ_p. We can also assume that if R is such that $\hat{R}\cap\{A_{-p}\cup A_p\} = \varnothing$ $\zeta(R)$ does not depend on p and $\bar{\zeta}(R') = \bar{\zeta}(R)$ if also $\hat{R}'\cap\{A_{-p}\cup A_p\} = \varnothing$, and R is obtained from R by a translation of an integer multiple of $(n+1)L$.

The $\bar{\zeta}$'s of course satisfy the estimate (3.35). The former remarks permit us to express the complex valued partition function in the volume Λ_p as

$$\begin{aligned}Z_{\Lambda_p}(\omega) &= \lim_{N\to\infty} Z_{\Lambda_p,N}(\omega)\\ &= \lambda_L^{2p(n+1)+1}\left[v\exp(-h_L/2), \chi_M\lambda_L^{-1/2}\right]^2(v\tilde{v}, \chi_M)^{2p-1}\\ &\quad\times\left[1 + \sum_{k\geqslant 1}\ \sum_{\substack{R_1,\ldots,R_k\\ \hat{R}_i\subset\Lambda_p\\ \hat{R}_i\cap\hat{R}_j=\varnothing,\ i\neq j}}\ \prod_{i=1}^{k}\bar{\zeta}(R_i)\right]\end{aligned} \tag{3.37}$$

The first equality in (3.37) follows from the Lebesgue theorem, the last one is obtained from (3.11) by letting N tend to infinity through the sequence of N_i.

The proof of the analyticity follows now straightforwardly from the decomposition (3.37) and the estimates of the activities of the polymers given in (3.35), by the method of the cluster expansion, that is based on the following lemma:

Lemma 3.2. Let

$$\Xi_{\Lambda_p}(\omega) = 1 + \sum_{k\geqslant 1}\ \sum_{\substack{R_1,\ldots,R_k\\ \hat{R}_i\subset\Lambda_p,\ \hat{R}_i\cap\hat{R}_j=\varnothing,\ i\neq j}}\ \prod_{i=1}^{k}\bar{\zeta}(R_i) \tag{3.38}$$

and

$$K = \sum_{\substack{C \in \mathscr{C}_p \cup \mathscr{P}_p \\ C \ni A_0}} g_C + 4g_{\{A_0, A_1\}}$$

For any given σ, $1/2 \leqslant \sigma < 1$ n, L, ω_0 can be chosen in such a way that, for every $\omega < \omega_0$,

(i) $$\exp K < 1/\left[\sqrt{\sigma}\,(2-\sqrt{\sigma})\right] \tag{3.39}$$

and, uniformly in p, if $|\bar{\zeta}(R)| \leqslant \sigma^{|R|}\prod_{C \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4} g_C$

(ii) $$\sup_{V \in \{A_i,\, -p \leqslant i \leqslant p\} \cup \{B_i,\, -p \leqslant i \leqslant p-1\}} \sum_{R\,:\, V \subset \hat{R} \subset \Lambda_p} |\bar{\zeta}(R)|$$

$$\leqslant \sigma K\left[1 - (e^K - 1)/(1 + \sigma^2 e^K - 2\sigma e^K)\right] \equiv G(K,\sigma) \tag{3.40}$$

(iii) $$\sum_{k \geqslant 1} \sum_{\substack{R_1, \ldots, R_k \\ R_i = R \text{ for some } i}} |\varphi_T(R_1, \ldots, R_k)| \prod_{i=1}^{k} |\bar{\zeta}(R_i)|$$

$$\leqslant \bar{\zeta}(R)\exp\left[G(K,\sqrt{\sigma})|R|\right]/\left\{1 - \sqrt{\sigma}\exp\left[G(K,\sqrt{\sigma})\right]\right\} \tag{3.41}$$

where

$$\varphi_T(R_1, \ldots, R_k) = \frac{1}{k!} \sum_{g \in \mathscr{G}_k(R_1, \ldots, R_k)} (-1)^{\#\{\text{edges in } g\}}$$

and $\mathscr{G}_k(R_1, \ldots, R_k)$ is the set of all connected subgraphs of the graph with vertices $\{1, \ldots, k\}$ and edges (i, j) corresponding to those pairs R_i, R_j such that $\hat{R}_i \cap \hat{R}_j \neq \emptyset$ and the sum is set to be equal to 0 if $\mathscr{G}_k$ is empty, and to 1 if $k = 1$;

(iv) $$\Xi_{\Lambda_p}(\omega) = \exp\left[\sum_{k \geqslant 1} \sum_{R_1, \ldots, R_k} \varphi_T(R_1, \ldots, R_k) \prod_{i=1}^{k} \bar{\zeta}(R_i)\right] \tag{3.42}$$

Proof. In Ref. 7 (see Lemma 1) it is proven that (i) implies (ii), (iii), (iv). (i) is a straightforward consequence of the definition of g_c. Indeed, for given $1/2 \leqslant \sigma < 1$, we can choose M sufficiently large, then $\bar{n}(M)$, $\bar{L}(n, M), \bar{\omega}(n, M, L)$ such that by choosing $n > \bar{n}$, $L > \bar{L}(n, M)$, $\omega < \bar{\omega}(n, M, L)$, K can be made arbitrarily small. ∎

We can now prove Theorem 1.1.

Proof of Theorem 1.1. Using (3.37) and (3.42) if M, n, L, ω are such that (3.39) holds we get

$$\begin{aligned}\ln Z_{\Lambda_p}(\omega) = &\left[2p(n+1)+1\right]\ln\lambda + 2\ln\left[v\exp(-h/2), \chi_M\lambda^{-1/2}\right] \\ &+ (2p-1)\ln(v\tilde{v}, \chi_M) \\ &+ \sum_{k\geqslant 1}\sum_{\substack{R_1,\ldots,R_k \\ \hat{R}_i\subset\Lambda_p}} \varphi_T(R_1,\ldots,R_k)\prod_{i=1}^{k}\bar{\zeta}(R_i)\end{aligned} \tag{3.43}$$

We note that the first three terms on the right-hand side of (3.43) do not depend on ω and, divided by $|\Lambda_p|$, converge to a limit as p tends to infinity. So we need only to consider the last term on the right-hand side of (3.43).

We shall denote by $\tilde{\zeta}(R)$ the activity of R computed in a volume Λ_p so large that $\hat{R} \cap \{A_{-p} \cup A_p\} = \emptyset$. It is clear that $\tilde{\zeta}(R)$ does not depend on p, and is invariant under translations of multiples of $(n+1)L$.

Given p, let us denote by Λ'_p the set $\Lambda_p \backslash \{A_{-p} \cup A_p\}$. We can decompose the last term on the right-hand side of (3.43) in the following way:

$$\begin{aligned}&\sum_{k\geqslant 1}\sum_{\substack{R_1,\ldots,R_k \\ \hat{R}_1\subset\Lambda_p}} \varphi_T(R_1,\ldots,R_k)\prod_{i=1}^{k}\bar{\zeta}(R_i) \\ &\quad= \sum_{k\geqslant 1}\sum_{\substack{R_1,\ldots,R_k \\ \hat{R}_i\subset\Lambda'_p}} \varphi_T(R_1,\ldots,R_k)\prod_{i=1}^{k}\tilde{\zeta}(R_i) \\ &\quad\quad+ \sum_{k\geqslant 1}\sum_{\substack{R_1,\ldots,R_k \\ \bigcup_{i=1}^{k}\hat{R}_i\cap\{A_{-p}\cup A_p\}\neq\emptyset}} \varphi_T(R_1,\ldots,R_k)\prod_{i=1}^{k}\bar{\zeta}(R_i)\end{aligned} \tag{3.44}$$

It follows from Lemma 3.1, that, choosing M, n, L, ω_0 such that K is sufficiently small, for every $\omega < \omega_0$ we have:

$$\exp K < 1/\left[\sigma e^{G(K,\sqrt{\sigma})}\left(2 - \sigma e^{G(K,\sqrt{\sigma})}\right)\right] \tag{3.45}$$

and then, if $V \in \{A_i, -p \leqslant i \leqslant p\} \cup \{B_i, -p \leqslant i \leqslant p-1\}$

$$\begin{aligned}&\sum_{k\geqslant 1}\sum_{\substack{R_1,\ldots,R_k \\ \bigcup_{i=1}^{K}\hat{R}_i\cap V\neq\emptyset}} \left|\varphi_T(R_1,\ldots,R_k)\prod_{i=1}^{K}|\tilde{\zeta}(R_i)\right| \\ &\quad\leqslant \sum_{R\,:\,\hat{R}\cap V\neq\emptyset} |\tilde{\zeta}(R)|\exp\left[G(K,\sqrt{\sigma})|R|\right]\Big/\left\{1-\sqrt{\sigma}\exp\left[G(K,\sqrt{\sigma})\right]\right\} \\ &\quad\leqslant G\left(K, \sigma e^{G(K,\sqrt{\sigma})}\right)\Big/\left\{1-\sqrt{\sigma}\exp\left[G(K,\sqrt{\sigma})\right]\right\}\end{aligned} \tag{3.46}$$

Of course (3.46) is still true if we replace $\tilde{\zeta}$ with $\bar{\zeta}$ for every Λ_p.

Using (3.46), we see then that the second term on the right-hand side of (3.44) does not contribute to the pressure. Exploiting then the translational invariance of $\tilde{\zeta}$ we get

$$\lim_{p\to\infty} \frac{\ln Z_{\Lambda_p}(\omega)}{|\Lambda_p|}$$

$$= \frac{\ln \lambda_L}{L} + \frac{\ln(v\tilde{v}, \chi_M)L}{n+1}$$

$$+ \frac{1}{n+1} \sum_{k\geq 1} \sum_{\substack{R_1, \ldots, R_k \\ \bigcup_{i=1}^k \hat{R}_i \cap \{A_0 \cup B_0\} \neq \emptyset \\ \bigcup_{i=1}^k \hat{R}_i \subset [(L-1)/2, +\infty]}} \varphi_T(R_1, \ldots, R_k) \prod_{i=1}^k \tilde{\zeta}(R_i) \quad (3.47)$$

It follows then from (3.47) that the pressure is an analytic function of ω, for $\omega < \omega_0$, as uniformly convergent series of analytic functions.

We have taken the limit on the particular sequence $\{\Lambda_p\}$ just for the sake of simplicity, and the same result is true for an arbitrary increasing sequence of intervals tending to $\mathbb{Z}$.

4. CONCLUDING REMARKS

We discuss here some possible extensions of our results.

(1) *Correlation functions.* The analyticity of the correlation functions in terms of the interaction parameters can be proven starting from the cluster expansion of the partition function. Indeed it is easily seen that we do not need the translational invariance of the complex perturbations, but just some uniform bounds of their strength, like those given in (1.8), (1.9), in order to express the complex partition function as an exponential of a convergent series of local quantities. Then, since the correlation functions can be seen as ratios between partition functions with locally perturbed Hamiltonians, it is clear that we can prove uniform estimates for these quantities by using the polymer representation. These estimates are still valid in the thermodynamical limit and the analyticity follows. Clustering properties of the correlation functions are also easy to obtain.

(2) *Boundary conditions.* In the previous sections only the case of empty boundary conditions was considered. Our proof, however, applies with minor changes to boundary conditions such that the values of the "spins" in the conditioning configurations do not increase "too fast" with

the distance from Λ_p. Here "not too fast" depends on the rate of decay of the interactions.

(3) *Uniqueness*. Our results imply uniqueness of the Gibbs measures for a much larger class of potential than those considered in Ref. 6. There the author considered continuous systems of particles on the line interacting either via positive potential with a more than exponential decay or via superstable finite-range potentials.

(4) *Many-body potentials*. Many-body potentials with an exponential decay in the number of interacting particles or spins can also be treated along the same lines of Ref. 7.

(5) *Divergent potentials*. In the case of particles we can consider superstable potentials positively diverging at short distances and complex perturbations exhibiting at most the same divergence, by proving, for example, the analyticity in the temperature. Here we need some more probability estimates in order to control the collapses of particles.

(6) *Many-dimensional systems in strips*. It is clear that our methods apply also to the case of many-dimensional systems of particles or unbounded spins confined in strips of the type $[-a,a]^{\gamma-1}\times\mathbb{R}$ or $[-a,a]^{\gamma-1}\times\mathbb{Z}$. Indeed these systems can be reduced to strictly one-dimensional systems by taking a suitable state space and our results apply also to this state space.

APPENDIX A

In this Appendix we shall prove P.1, P.2, and P.3 of Section 1.

In force of the hypotheses on the potential, the operator $T_{N,L}$ is compact and it is strongly positive in the following sense: Given the cone

$$K=\left\{u\in\mathscr{C}\big((\mathfrak{X}_{\leqslant N})^L\big),\, u(s)\geqslant 0\right\}$$

and, calling K^0 its interior

$$K^0=\left\{u\in\mathscr{C}\big((\mathfrak{X}_{\leqslant N})^L\big),\, u(s)>0\right\}$$

then

$$T_{N,L}(K\setminus\{0\})\subset K^0$$

P.1 follows then from a theorem by Krein and Rutman (Ref. 13, p. 267, Theorem 6.3) and from the symmetry property (1.18). In order to prove P.2

and P.3, we remark that, as a consequence of P.1,

$$\lim_{k\to\infty} T^k_{N,L}\exp(-h_L/2)/\lambda^k_{N,L} = \left\{\int \nu^N_L(ds)\exp\left[-h_L(s)/2\right]\tilde{v}_{N,L}(s)\right\} v_{N,L}$$
$$\lim_{k\to\infty} \exp(-h_L/2)T^k_{N,L}/\lambda^k_{N,L} = \left\{\int \nu^N_L(ds)\exp\left[-h_L(s)/2\right]v_{N,L}(s)\right\} \tilde{v}_{N,L} \tag{A1}$$

where the limit is intended in the topology of the uniform convergence.

It follows from (A1) that

$$\lambda_{N,L} = \lim_{k\to\infty}\left\{\int \nu^N_L(ds)\exp\left[-h_L(s)/2\right]T^k_{N,L}(s,s') \times\exp\left[-h_L(s')/2\right]\nu^N_L(ds')\right\}^{1/k}$$
$$= \lim_{k\to\infty}\left(\tilde{Z}_{\Lambda_p,N}\right)^{L/(|\Lambda_p|-L)} \tag{A2}$$

For $N_1 < N_2$, $\tilde{Z}_{\Lambda_p,N_1} < \tilde{Z}_{\Lambda_p,N_2}$; hence, for every L, the sequence $\lambda_{N,L}$ is not decreasing in N. From the superstability hypotheses (1.4) and from (1.3) it follows that there exists L_0 such that for $L > L_0$ and any $\Lambda \subset \mathbb{Z}$:

$$H^{<L}_\Lambda(s_\Lambda) \geqslant \sum_{x\in\Lambda}\left(\frac{A}{2}|s_x|^2 - \delta\right) \tag{A3}$$

Then for any $L > L_0$ and any $N > 0$

$$\tilde{Z}_{\Lambda_p,N} = \int \nu^N_{\Lambda_p}(ds_{\Lambda_p})\exp\left[-H^{<L}_{\Lambda_p}(s_p)\right] \leqslant \left\{\int \nu(ds)\exp\left[-\left(\frac{A}{2}|s|^2-\delta\right)\right]\right\}^{|\Lambda_p|} \tag{A4}$$

This proves that, for $L > L_0$, $\lambda_L = \lim_{N\to\infty}\lambda_{N,L}$ exists and satisfies

$$\lambda_L \leqslant \left\{\int \nu(ds)\exp\left[-\left(\frac{A}{2}|s|^2-\delta\right)\right]\right\}^L \tag{A5}$$

Let now the correlation functions of the block model with only nearest-neighbor block interaction be defined as

$$\rho_{\Lambda_p,N}(s_\Delta) = \int \nu^N_{\Lambda_p\setminus\Delta}(ds_{\Lambda_p\setminus\Delta})\exp\left[-H^{<L}_{\Lambda_p}(s_{\Lambda_p})\right]/\tilde{Z}_{\Lambda_p,N} \tag{A6}$$

Following the proofs of Proposition 2.7 of Ref. 9 and Theorem 2.2 of Ref. 10, it is easy to check that there exist N_0, A^* and δ^* such that, for every $N > N_0$, $L > L_0$, Δ, p with $\Delta \subset \Lambda_p$,

$$\rho_{\Lambda_p,N}(s_\Delta) \leqslant \exp\left[-\sum_{x\in\Delta}\left(A^*|s_x|^2 - \delta^*\right)\right] \tag{A7}$$

P.3 is then a straightforward consequence of (A1) and (A7).

We also remark that it follows from P.3 and Proposition 2.2 that one can choose $\overline{M}$ such that

$$\mu\left((\mathfrak{X}_{\leqslant N})^{KL}\right) \leqslant \sup_{s\in\mathscr{S}_{\overline{M}}} \left[u(s)/v(s)\right] \sup_{s\in\mathscr{S}_{\overline{M}}} \left[\tilde{u}(s)/\tilde{v}(s)\right]$$

$$\times \frac{\int \nu_L^N(ds)\,\tilde{u}(s)\,T_{N,L}^{k-1}(s,s')u(s')\nu_L^N(ds')}{\int \nu_L^N(ds)\,\tilde{u}(s)\chi_{\overline{M}}(s)\,T_{N,L}^{k-1}(s,s')u(s')\chi_{\overline{M}}(s')\nu_L^N(ds')}$$

$$\leqslant \left(\tfrac{4}{3}\mathscr{N}\right)^2 \tag{A8}$$

where $\mathscr{N}$ does not depend on N, L, and k.

APPENDIX B

In this Appendix we prove the following:

Proposition B.1. Let $M > 1$ and let the sequence $(a_k)_{k\geqslant 1}$ satisfy

(i) $a_k \geqslant 0$

(ii) $a_1 \geqslant e^{-d_1 M^2}$

(iii) $\sum_{k=1}^{\infty} a_k = 1$

(iv) $a_k \leqslant \exp\left(d_2 M^2 - k e^{-d_3 M^2}\right)$

where d_1, d_2, d_3 are strictly positive constants.

Let the constant σ and the sequence $(b_k)_{k\geqslant 0}$ be defined by

$$\sigma = \sum_{k=1}^{\infty} k a_k \tag{B1}$$

$$\left(1 - \sum_{k=1}^{\infty} a_k s^k\right)^{-1} = \sum_{k=0}^{\infty} b_k s^k \qquad \text{for} \quad |s| < 1 \tag{B2}$$

Then there exist $d_1', d_2' > 0$, depending only on d_1, d_2 and d_3, such that

$$|b_k - \sigma| \leqslant \exp\left(d_1' M^2 - k e^{-d_2' M^2}\right) \tag{B3}$$

It is clear by simple probabilistic considerations and by (2.23), (2.24) that we can apply Proposition B.1 with appropriate constants d_1, d_2, and d_3 by taking $a_k = e\Pi_{ee}^k$, $b_k = \Pi_{ee}^k$ and $\sigma = \pi_e$.

Proof. Let us define $\Phi(s)$ as

$$\Phi(s) = 1 - \sum_{k=1}^{\infty} a_k s^k \tag{B5}$$

we have $\Phi(1)=0$, $\Phi'(1)=-\sum_{k=1}^{\infty}ka_k=-\sigma^{-1}$. $\Phi(s)^{-1}$ has a pole of order 1 in $s=1$.

We have $\mathrm{Res}_{s=1}\Phi(s)^{-1}=\Phi'(1)^{-1}=-\sigma$.

Now let us define

$$F(s)=\phi(s)/(s-1)=\sum_{k=0}^{\infty}c_k s^k \tag{B6}$$

We have

$$c_k=-\left(1-\sum_{l=1}^{k}a_l\right)=-\sum_{l=k+1}^{\infty}a_l$$

and, from (iv),

$$|c_k|\leqslant \alpha e^{-\beta(k+1)}/(1-e^{-\beta}) \tag{B7}$$

where $\alpha=e^{d_2M^2}$, $\beta=e^{-d_3M^2}$. Therefore, for $|s|<e^{\beta}$

$$|F'(s)|=\left|\sum_{k=1}^{\infty}kc_k s^k\right|\leqslant \alpha|s|e^{-2\beta}(1-e^{-\beta})^{-1}(1-|s|e^{-\beta})^{-2} \tag{B8}$$

Let $s=re^{i\theta}$ with $r<\min(e^{\beta/2},2)$

$$\begin{aligned}|F(s)|&=\left|F(1)+i\int_0^{\theta}F'(e^{i\tau})e^{i\tau}\,d\tau+e^{i\theta}\int_1^{r}F'(te^{i\theta})\,dt\right|\\&\geqslant 1-|\theta|\left[\alpha e^{-2\beta}(1-e^{-\beta})^{-3}\right]\\&\quad-\frac{r^2-1}{2}\alpha e^{-(3/2)\beta}(1-e^{-\beta})^{-1}(1-e^{-\beta/2})^{-2}\\&\geqslant 1-(|\theta|+|r-1|)\frac{3}{2}\alpha(1-e^{-\beta/2})^{-3}\end{aligned}$$

Then $|F(s)|\geqslant\frac{1}{2}$ if $\max(|\theta|,|r-1|)\leqslant\frac{2}{6}(1-e^{-\beta/2})^3\alpha=\gamma$. Now let $|s|=1$, $s=e^{i\theta}$, $\theta\in[-\pi,\pi]$, $|\theta|>\gamma$.

$$\begin{aligned}|\Phi(s)|&=\left|1-\sum_{k=1}^{\infty}a_ks^k\right|\geqslant\left|\mathrm{Re}\left(1-\sum_{k=1}^{\infty}a_ks^k\right)\right|\\&=\left|a_1(1-\cos\theta)+\sum_{k\geqslant 2}a_k(1-\cos k\theta)\right|\geqslant a_1(1-\cos\theta)\end{aligned}$$

so that if $s=re^{i\theta}$, $1\leqslant r<\min(e^{\beta/2},2)$, $\theta\in[-\pi,\pi]$, $|\theta|>\gamma$, we have

$$\begin{aligned}|\Phi(s)|&=\left|\Phi(e^{i\theta})+e^{i\theta}\int_1^{r}\Phi'(te^{i\theta})\,dt\right|\\&\geqslant a_1(1-\cos\theta)-\tfrac{3}{2}(r-1)\alpha e^{-\beta/2}(1-e^{-\beta/2})^{-2}\end{aligned}$$

Therefore, if $r-1 \leqslant \frac{2}{3}(1-e^{-\beta/2})^2 a_1(1-\cos\gamma)$,

$$|\Phi(s)| \geqslant a_1(1-\cos\theta)/2$$

Let

$$R = \min\left(\tfrac{1}{3}(1-e^{-\beta/2})^3/\alpha, \tfrac{2}{3}(1-e^{-\beta/2})^2 a_1(1-\cos\gamma), e^{\beta/2}, 2\right) \quad \text{(B9)}$$

and let $|s| = R$, $s = Re^{i\theta}$, $\theta \in [-\pi, \pi]$. If $|\theta| > \gamma$, then

$$\left|\frac{1}{\Phi(s)} - \frac{\sigma}{1-s}\right| \leqslant \frac{1}{|\Phi(s)|} + \frac{\sigma}{|s-1|} \leqslant \frac{2}{a_1(1-\cos\gamma)} + \frac{1}{(1-\cos\gamma)^{1/2}} \quad \text{(B10)}$$

If $|\theta| \leqslant \gamma$

$$\left|\frac{1}{\Phi(s)} - \frac{\sigma}{1-s}\right| = \left|\frac{1+\sigma F(s)}{s-1}\right| \cdot \frac{1}{|F(s)|} \leqslant 2\left|\frac{1+\sigma F(s)}{1-s}\right|$$

On the other hand,

$$\frac{1+\sigma F(s)}{1-s} = \left(1 + \sum_{k=0}^{\infty} \sigma c_k s^k\right)\left(\sum_{l=0}^{\infty} s^l\right) = \sum_{k=0}^{\infty}\left(1 + \sigma \sum_{l=0}^{k} c_l\right) s^k$$

$$= \sum_{k=0}^{\infty}\left(-\sigma \sum_{l=k+1}^{\infty} c_l\right) s^k = \sum_{k=0}^{\infty} d_k s^k$$

(where we have used the fact that $\sum_{l=0}^{\infty} c_l = -\sum_{k=1}^{\infty} k a_k = -\sigma^{-1}$) with $|d_k| \leqslant \alpha e^{-2\beta}(1-e^{-\beta})^{-2} e^{-\beta k}$. Therefore for $s = Re^{i\theta}$, $|\theta| \leqslant \gamma$:

$$\left|\frac{1}{\Phi(s)} - \frac{\sigma}{s-1}\right| \leqslant 2\alpha e^{-2\beta}(1-e^{-\beta})^{-2} \sum_{k=0}^{\infty} R^k e^{-\beta k}$$

$$\leqslant 2\alpha e^{-2\beta}(1-e^{-\beta})^{-2}(1-e^{-\beta/2})^{-1} \quad \text{(B11)}$$

From (B10), (B11) we have that if $|s| = R$,

$$\left|\frac{1}{\Phi(s)} - \frac{\sigma}{s-1}\right|$$

$$\leqslant \max\left[\frac{1}{a_1(1-\cos\gamma)} + \frac{1}{(1-\cos\gamma)^{1/2}}, \frac{2\alpha e^{-2\beta}}{(1-e^{-\beta})^2(1-e^{-\beta/2})}\right]$$

$$\equiv \delta \quad \text{(B12)}$$

Now, since

$$\frac{1}{\Phi(s)} - \frac{\sigma}{s-1} = \sum_{k=0}^{\infty}(b_k - \sigma)s^k,$$

we obtain, denoting by C_R the circle with center O and radius R,

$$|b_k - \sigma| = \left| \frac{1}{2\pi i} \oint_{C_R} \left(\frac{1}{\Phi(s)} - \frac{\sigma}{s-1} \right) s^{-(k+1)}\, ds \right|$$

$$\leqslant \frac{1}{2\pi} 2\pi R \max_{|s|=R} \left| \frac{1}{\Phi(s)} - \frac{\sigma}{s-1} \right| R^{-(k+1)}$$

$$\leqslant \delta R^{-k}$$

that concludes the proof. ■

NOTE

After this paper was submitted, J. Lebowitz made us aware of the existence of an unpublished manuscript by O. Lanford, in which he proves the analyticity of the free energy for one-dimensional systems of particles with finite-range superstable interaction and the uniqueness of the Gibbs state and the continuous differentiability of the free energy for infinite-range superstable potentials decaying sufficiently fast [essentially as $r^{-2}(\ln r)^{-(2+\epsilon)}$].

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